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Aids to Computation

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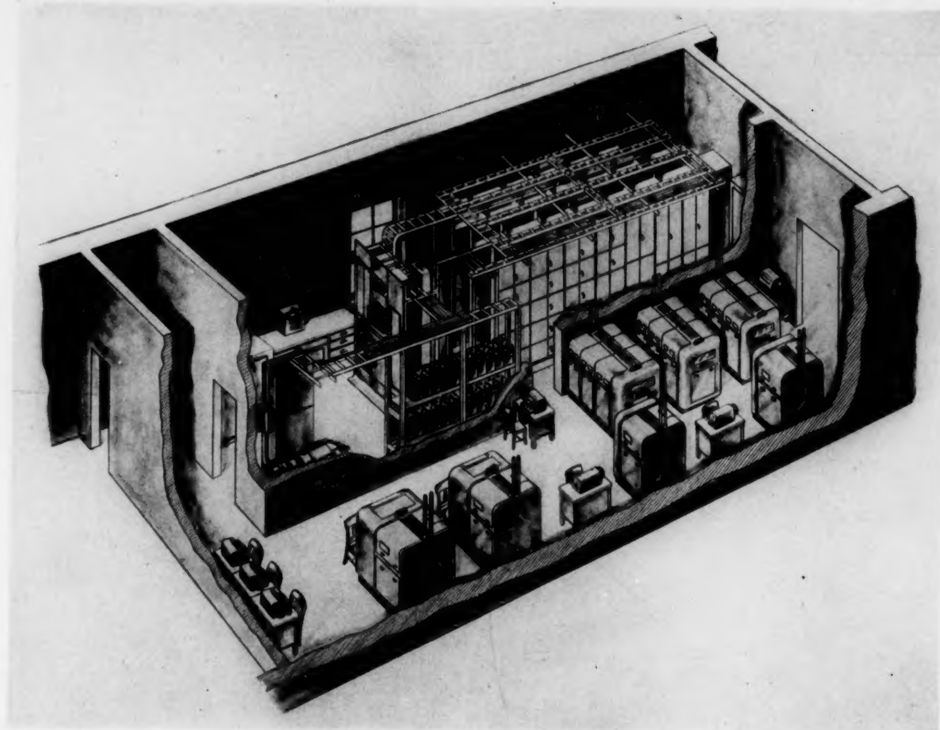

A Bell Telephone Laboratories' Computing Machine

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A Bell Telephone Laboratories' Computing Machine—I

This is the first of two articles by Dr. Alt describing a Bell Telephone Laboratories' machine. The second article will treat machine control, operational characteristics, and problems for the machine.

1. Introduction

High-speed digital computing machines which have been developed during the past few years are of two kinds, electronic and electro-mechanical. In the latter class two families have come to prominence. One consists of the machines built by H. H. Aiken at Harvard University,[1] the other of a series of machines built by the Bell Telephone Laboratories in New York. This article deals with the latter family, and specifically with its largest and youngest member, a machine which has been built in the past two years in duplicate, one unit for the National Advisory Committee for Aeronautics (Langley Field, Va.) and one for the Ordnance Department of the U. S. Army (Aberdeen Proving Ground, Md.).

Before entering upon a description of this machine, a brief enumeration of its "ancestors" may be in order. The basic idea underlying all machines in this family, that of using telephone switching equipment for computing purposes, was conceived several years before the outbreak of the last war by George R. Stibitz, then on the staff of the Bell Telephone Laboratories. The Company decided to test the idea on a small scale by building, for its own use, a machine capable of performing addition, subtraction, multiplication and division of complex numbers. Under the impact of the war-time demand for large-scale computing, the Company next developed the "Relay Interpolator," a machine consisting mainly of about 500 telephone relays, together with some teletype equipment used for transmitting numbers into and out of the machine and for directing the operations. The next machine to be built was the "Ballistic Computer," containing about 1300 relays, more elaborate and more complex than the Relay Interpolator but still, like all the earlier machines, a special-purpose device, designed to carry out only a few special kinds of computations. Finally, in 1944 the Bell Telephone Laboratories undertook to develop the all-purpose computing system with which this article is concerned. This system differs from its predecessors in size—it contains over 9000 relays and about 50 pieces of teletype apparatus, covers a floor space of about 1000 square feet, and weighs about 10 tons—as well as in flexibility, generality of application, reliability, and ability to operate automatically without requiring the presence of human operators.

When the complete description of this machine is written, including instructions for the coding of problems and the operation of the machine, description of all the circuits involved, specifications for the components and instructions for their maintenance, it will fill hundreds of pages. This article will be limited for the most part to those aspects which are of interest to mathematicians in general, rather than to persons operating the machine. It will therefore merely attempt to give a first idea of the methods by which the machine solves computing problems, to describe its capabilities and its limitations, and to make a few comparisons between this machine and other

large-scale computing machines, notably the Harvard machines mentioned above, the punch-card machines of the IBM Corporation, the ENIAC,[2] and some of the projected electronic machines of the future.

The author has had the benefit of informal discussion with H. H. Aiken, E. G. Andrews, J. von Neumann, S. B. Williams, and many others, too numerous to mention, all of whom have greatly contributed to crystallizing the views set forth in this article.

2. Number Storage. The Bi-quinary System

a. Number Systems.—Every calculating machine stores numbers by setting some elements (such as wheels, shafts, vacuum tubes, or the like) in one of several possible (stable) states. This machine uses relays, which are capable of being maintained in either of two stable states: a relay may be operated ("up") or released ("down"). If operated, it closes a group of electric paths and opens another group; if released, it opens the former group and closes the latter. Each closed path may be used, if desired, to operate other relays.

Since each relay has two stable states, a natural way to apply relays to number storage would be to convert numbers from the decimal to the binary system and store them in the latter form. This method was not used. Instead, numbers are represented in a system called the "bi-quinary system." Each decimal digit is replaced by two digits, of which one (the "quinary" digit) has one of the values from 0 to 4 and the other (the "binary" digit) has one of the two values 0 or 5, such that the sum of the two values is equal to the value of the decimal digit which is to be represented. There are two relays to store each binary digit, and five relays to store each quinary digit; when a number is stored, exactly one relay in each group is operated. There are thus seven relays required for each decimal digit stored. This is a saving as compared with the most obvious system of using ten elements, one for each of the ten values of a decimal digit. (This latter system is in use, for instance, on the ENIAC.) It is, on the other hand, wasteful by comparison with binary system as well as with some other systems that have been devised. For instance, the recently developed "Relay Multipliers" of the IBM Corporation use the bi-quinary system, described above, with the modification that only one relay is used to store the binary digit (the value 0 being stored by having that relay released), and four relays are used to store the quinary digit (the value 0 being stored by having all four released). The Harvard "Mark II" translates every decimal digit into four binary digits and stores these on four elements; this system differs from the straight binary system in that it translates each decimal digit separately, whereas the binary system translates the entire number with all its digits into binary form. The system originally suggested by Stibitz consisted in representing a decimal digit of value d on four relays by expressing the number $d + 3$ in the binary system; this arrangement simplifies the design of the calculator. Various other systems of number representation have been under discussion.

9's Complements
2 - 0101
7 - 1010
9 | 1111

b. Relative Merits of Various Systems.—The question of which system of representation is to be used comes up in the design of any computing machine which uses an inherently binary storage device, such as relay or vacuum tube, i.e., a device capable of maintaining two stable states. It does not exist, for example, in most desk-type machines and most punch-card

machines, which use counter wheels for number storage. All these latter machines use the usual decimal system.

The designers of the bi-quinary system acknowledge that their system requires more storage elements per digit stored than some other systems, but maintain that this drawback is more than compensated for by two advantages: no special equipment is needed to translate the numbers put into the machine from the decimal system to the system used in the machine, or to perform the opposite translation on the output of the machine (the translation between decimal and bi-quinary system is trivial); and the system can be made self-checking, so that a stored number cannot be changed into another number by some transient machine trouble. Checking is accomplished by means of a circuit which is closed when exactly one binary and exactly one quinary relay are "up" for every digit stored, and which is open under all other conditions.

These advantages are extremely important and are well worth the price of additional storage elements needed; but there are systems that would have maintained both advantages and saved at least some of the excessive storage equipment. For example, each decimal digit could be represented on a group of five relays, of which exactly two are operated when a digit is stored. There are exactly ten combinations of two out of five elements, and these can be made to correspond to the ten values of a decimal digit. (Combinations of three out of five elements can be used in the same way, and are, in fact, used in a few places in the machine which we are discussing.) A checking circuit can be devised which is closed only when exactly two relays are up, is open in every other case, and stops the machine when open. This gives the same degree of insurance against error as the bi-quinary system: an error will pass only when a failure of a relay is offset by another relay in the same group being falsely operated. Experience shows that the probability of such an occurrence is remote.

It is desirable to arrange a computing machine in such a way that it stops at once when any part of its equipment runs into trouble. Such an arrangement greatly facilitates the task of locating the source of trouble, a formidable task even under the most favorable circumstances, as anyone knows who has tried it. Our machine fulfills this requirement for the most part, although not entirely. For example, in the case of number storage, any "over-registration" (i.e., any case of more relays being operated than are needed for storing a number) will halt the machine instantly. But an under-registration will stop it only at the time when the number is being transferred into a register or when a stored number is called for. This is, however, one of a small number of cases in which the machine does not halt at once in case of trouble, and it has not been the cause of any undue difficulty in operating the machine.

c. Digital Capacity and the Floating Decimal Point.—As has been said above, each decimal digit of a number is stored on seven relays. The sign of a number, + or −, is stored on two relays, exactly one of which is operated whenever a number is stored. Negative numbers are represented by the minus sign and the absolute amount of the number, not by its complement as is the case on some other machines.

The standard length of numbers stored in the machine is seven significant decimal digits. For comparison, the length of numbers stored on some other

computing machines is as follows: desk machines, usually 8 or 10; standard IBM punch-card machines, up to 8; IBM Relay Multipliers, 6, and for certain operations, 12; H. H. Aiken's "Mark I", 23; "Mark II," 10; ENIAC, 10.

These figures are not strictly comparable, for at least two reasons. One is that, as machines become faster, they will presumably do longer problems in which the accumulated effect of rounding errors becomes more pronounced. Therefore, the faster a machine, the more digits should it provide. The other reason is that for two of the machines listed the numbers given refer to significant digits, while for all the other machines all numbers stored are limited to a given range of decimal positions. The ability of a machine to retain only significant digits and to discard the insignificant (zero) digits in front of a number is sometimes referred to as "floating decimal point."

The Bell Laboratories' machine was the first to use a floating decimal point. The only other existing machine which uses it is the "Mark II." In our machine it works as follows: each number is transformed into a decimal number with seven significant digits, of which the first is immediately to the right of the decimal point, multiplied by an appropriate power of ten. The machine stores the sign of the number, the first seven decimal digits, and the exponent of the power of ten by which the decimal is to be multiplied. (This exponent is briefly called the "exponent of the number.") Exponents are limited to the range from -19 to $+19$.

The "exponent of a number" on our machine consists of a sign ($+$ or $-$), a first digit which is limited to the value 0 or 1, and a second digit between 0 and 9. The sign and tens digit are stored on one of two relays each, and the units digit on two out of seven relays in the bi-quinary system. Together with the 49 relays needed to store the seven significant digits of the number and the two relays needed to store its sign, a total of 62 relays are required to store one number. These, together with a few operating relays, form what is called a *register*. The machine contains 44 registers, of which 30 serve only to store numbers, while the remaining 14 perform various additional functions.

3. Calculation

a. Addition and Subtraction.—For adding two numbers, three groups of relays designated A, B and C, respectively, are provided in the machine. Each group is capable of storing a ten-digit number in the bi-quinary system (the first of the ten digits being limited to values 0, 1, 9). They are called "groups" rather than "registers" because of the absence of sign and exponent. The groups are wired together in such a way that if two numbers are stored on the A and B groups, those relays in the C group which represent the sum of the two numbers are automatically operated.

The details of these wirings are not of interest for our present purpose. The only point that may be interesting is the handling of carries. Each digit provides for two adding circuits: the one mentioned before, and another one which gives a sum higher by 1 (or 0 instead of 9) and which is used in case there is an incoming carry of 1. The addition of the two numbers is done in two steps which are carried out simultaneously. One sets up one of two conditions—incoming carry 0 or 1—for each digit. The other step carries out the addition (i.e., operates 2 relays in each digit of the C group) by using

one of the two circuits in accordance with the incarry condition. In the former step, the carry conditions are determined from a three-way distinction regarding the sums of the two digits in the next-lower position, as follows: sum less than 9, out-carry 0; sum more than 9, out-carry 1; sum equal to 9, out-carry equal to in-carry.

In addition to the circuits for adding A and B and putting the sum on C, described above, there is a set of wires which will similarly operate those of the B-relays which represent the sum of two numbers stored on the A and C relays.

If the machine receives an order to add two (positive) numbers, it first compares their exponents. The number with the greater exponent (in case of equal exponents, the first of the two numbers) is transferred to the A group of relays. Of the ten digits for which this group is equipped, those from the third to the ninth correspond to the seven significant digits of a storing register. The number with the smaller exponent is transferred to the B relays, after being shifted to the right by a number of decimal places equal to the difference between the two exponents. That is, if the difference between the two exponents is 1, the number is placed onto the fourth to tenth place of the B relays; if the difference of exponents is 2, the first six significant digits of the number are placed onto the fifth to tenth place of the B relays, while the seventh significant place is discarded; etc. The machine then closes the adding circuit mentioned above, which places the sum of the two numbers onto the C relays. The first seven significant digits of the sum (which may be located in digits 3 to 9 or in digits 2 to 8 of the C relays) are then transferred to a register designated for this purpose by the operator, after having first been rounded in the seventh place.

It will be noted that nine places in the A, B, and C relays would have been sufficient to accomplish this operation, namely, the seven places of the larger addend, plus one place on the left for carries and one on the right for rounding. The tenth was added mainly for simplification of the circuits, especially in connection with division.

To subtract two numbers, the machine converts the one with the larger exponent into its complement and stores this on the A relays. The other number is put onto the B relays, shifted as needed, and the sum appears on the C relays. This is the complement of the desired result, and the machine converts it back into a true number. If the exponents are equal, the first of the two numbers is converted to a complement and put on the A relays, regardless of the relative size of the two numbers. If the first happens to be smaller, the result is represented by its absolute value, rather than by its complement, and there will be a digit "0" in the left-hand position when the two numbers are added. The presence of this digit gives the signal to the machine to omit converting the result back to its complement, and to reverse the sign of the answer. The rounding and transferring out of the result occurs as in addition.

If one or both numbers involved in an addition or subtraction are negative, the machine determines from the combination of sign whether the absolute amounts should be added or substracted, and proceeds accordingly.

All complements are taken with respect to 9999999, and an "end around carry" into the right-hand position is simulated on addition in order to effect the necessary correction by 1 in this position.

If the result of an addition or subtraction has one or more zeros in the left-hand position—this happens if the two addends have opposite sign and agree in the first few places—the machine will automatically shift all digits of the answer to the left until the first significant digit occupies the left-hand position of the storage register, and will correct the exponent of the answer accordingly. This feature, briefly called "zero shift," can be canceled at the discretion of the operator. The rounding of the answer can also be canceled if desired. If the operator has not elected to cancel either the rounding or the zero shift, the machine will automatically omit rounding in case a zero shift is performed. This is necessary for obvious reasons. If the answer of a computation is zero, the calculator reads out a minus sign, seven zero digits, and an exponent of -19 (the smallest possible exponent within the capacity of the machine); if the operator has specified that the "zero shift" be canceled, the exponent read out is that of the larger of the two addends, and the sign may be either $+$ or $-$ depending on the signs of the addends.

b. Multiplication, Division, Square Root.—Multiplication is performed by repeated addition. The multiplicand is entered on the A relays, and the C relays are initially set to hold the number 0. By means of the adding circuits described above, the machine now adds A to C, records the sum on B and clears the C relays. It then adds A to B, records the result on C and clears B. It keeps on adding alternatingly $A + C$ onto B and $A + B$ onto C, the number of such additions being determined by the size of the first multiplier digit. After one digit of the multiplier is exhausted, the machine goes on to the next, and so on.

The reader will have noticed in the description of addition that the machine provides for two adding circuits, $A + B = C$ and $A + C = B$, of which only the first was used in addition and subtraction. The purpose of the second is to make possible the multiplication process just described. If a multiplier digit is 5 or more, the machine will perform repeated subtractions instead of additions (the number of subtractions being equal to the complement of the multiplier digit) supplemented by one addition in the next-higher decimal place.

Throughout those successive additions at least eight significant digits are retained. When the multiplication process is finished, any initial zeros will be shifted off just as in addition. As a rule there is at most one initial zero, but there may be more than one if one of the two factors had initial zeros. This condition may come about as a result of a previous operation in which the zero shift was canceled. If no zero shift is necessary, the answer will be rounded in the seventh place. This is accomplished automatically by adding a 5 in the eighth place and then dropping that place. Zero shift, rounding, or both may be canceled at the discretion of the operator.

After the multiplication is finished, the product is transferred to a register designated by the operator. The sign of the product is supplied by the machine in accordance with the signs of the two factors in the usual manner. The exponent of the product is determined by the machine as the sum of the exponents of the factors, corrected by the amount of any zero shift that may have been performed.

The result of the multiplication is accurate to seven significant digits (provided both factors had this accuracy). There is no possibility of utilizing

the subsequent digits of the product. It appears that it would have been possible and desirable to design the machine in such a way that fourteen digits of the product could be used if desired.

Just as multiplication is accomplished by repeated addition, so division and extraction of square roots are accomplished by repeated subtraction. We shall omit a detailed description of these processes here, since the general principles on which they are based are well enough exemplified by the case of multiplication. It may be mentioned that both processes are carried in the machine to seven significant digits, and that there is no possibility of utilizing the remainder after division or square root.

4. Input and Output

a. Input Systems.—The "input" for a computational problem (i.e., the information available before the start of the computation) consists of two kinds of elements: numbers, and "orders." The latter are the instructions as to the operations that are to be performed on the numbers.

The "output" or result of computation consists of numbers only. It has been proposed—facetiously by some, in earnest by others—to build a "thinking" machine whose output would be orders rather than numbers. From a small input such a machine would automatically produce the system of orders required for the solution of a complex problem on a computing machine. Perhaps such a machine will be developed in the future, but all existing machines produce only numbers.

An important characteristic of any computing machine is the method by which orders and numbers are transferred ("read") into the machine. It seems that most designers of computing machines agree that the time required for the input of one number should be of the same order of magnitude as the time required for an arithmetic operation, or perhaps longer by one order of magnitude. If the input time is outside these limits, an unbalanced machine results, because one component of the machine spends an unduly large proportion of its time waiting for another component. The input time for an order should be no longer than the time required to carry out the order. Punch-card machines are faster than desk machines and therefore need a faster input method. Such a method is realized in the feeding of cards, which proceeds at speeds in the neighborhood of ½ second per card (the exact speed varying with the type of machine), and is well in balance with the computing speed of these machines. Note that the speed is ½ second per card, not per number, and that one card usually holds several numbers. The Harvard machines use punched cards for most input data, and perforated tape for the rest. Again, the speed of input is well in balance with the computing speed. The ENIAC uses a card feed like that of standard punch-card machines. This is far out of balance with the computing speed of the machine; a computation is carried out in a few milliseconds, whereas reading of numbers requires ½ second.

The Bell Laboratories' machine uses teletype devices for reading and perforating of paper tape (with small modifications) for its final stage of input, both for numbers and orders. Its speed turns out to be approximately in balance with the computing speed of the machine, or perhaps slightly too slow. It takes about two seconds to read and transfer into the

machine either a (seven-digit) number or an order of average length. Additions and multiplications, on the other hand, can be carried out in 0.3 and 1.0 seconds, respectively. Thus, the input speed of numbers is adequate while that of orders is slightly too slow.

b. The Tape-Reading Process.—The tape used is $\frac{7}{8}$ inch wide and has room for six holes across the tape. These six holes are designated by the numbers 0 to 5. Decimal digits are represented on tape by combinations of three holes not including the hole numbered 0. There are exactly ten possible combinations of three out of the five holes 1 to 5, and these ten combinations represent the ten possible values of a decimal digit. Orders to the machine are written in a code consisting of the first twenty letters of the alphabet and these are represented on tape by all combinations of three holes out of six. (The representation of the first ten letters is identical with that of the ten decimal digits.) Some combinations of two or four holes are used for checking purposes; other two-hole combinations are used to represent the sign and exponent of a number; some combinations of four holes and most one- or five-hole combinations are used to designate special signals to the machine. Any such combination of between one and five holes, all lying in a row across the tape, is called a "code." Tapes are fed into standard teletype devices called "transmitters," which move the tapes and read one code at a time. The reading time is not determined by the mechanical construction of the transmitter but by the amount of work the machine does between codes. The transmitter moves the tape the distance between two codes; this requires about .05 seconds. Then the tape is stopped, a set of six sensing fingers comes up to the tape, and those fingers which find holes in the tape penetrate them, thereby closing certain electrical contacts and breaking others. The electric paths thus closed set up relays corresponding to the holes in the tapes. There follows a long chain of relay operations, until finally a signal is sent to the tape transmitter which causes it to advance the tape in preparation for reading the next code. For the purpose of this paper it is not necessary to enumerate all the steps performed, but it is important to realize that they happen serially, each step closing the electric path which will operate one or more relays in the next step. This chain of steps (many of which are introduced for the sole purpose of checking and safeguarding the proper functioning of the machine) is time-consuming and raises the total time elapsed between code readings to an average of .2 seconds. The exact reading time varies within wide limits, depending on the codes and on the context in which they come. There is nothing in the machine to synchronize the operations of its various parts; each relay operation engenders the next, just as fast as relays will click.

As a rule, eleven codes are needed to represent a seven-digit number with sign and exponent on tape. Orders vary in length, but an average is nine codes per order. Thus, the average reading time for both numbers and orders is in the neighborhood of 2 seconds.

c. Output.—The numerical output of the machine is recorded by perforating numbers on a tape and/or by printing on a sheet of paper. Both operations are performed by standard teletype devices called "reperforator" and "page printer" respectively.

When printing, appropriate orders must be given to the machine to determine the arrangement of numbers on a page, the number of decimal

places to be printed, etc. It is a little unfortunate that these orders have to be interspersed with the computing program, because this detracts from the general applicability of many programs. It might have been better to concentrate all the printing work in a separate auxiliary machine that would operate from tapes put out by the main machine. As it is, occasionally a program will have to be changed merely because the magnitude of the numbers involved changes and the printing orders have to be modified to accommodate more decimal places or the like.

The tape on which the results of the computation are perforated runs through a transmitter in which the numbers previously perforated may be read back into the machine if this is desired. This arrangement makes it possible to store intermediate results for later use in the same computation. In practice it turns out that this storage function is far more important than the function of holding answers. In fact, the machine could be improved by having more storage tapes. It seems to be the consensus of opinion that three storage tapes is a desirable minimum for any large computing machine.

d. Distribution of Input among Tapes.—The tapes which contain the input for any problem are classified into three groups called Problem, Routine, and Table Tapes. As a rule a problem may use up to five routine tapes (designated A to E), up to six table tapes (designated A to F), and one problem tape.

When the tape input system for this machine was designed, it was intended that the routine tapes should contain all the orders, the table tapes should contain numerical information of a general nature, comparable to function tables used in manual computing, and the problem tape should contain numerical information specific to the problem being solved (i.e., initial conditions for differential equations, coefficients of a system of algebraic equations, etc.). Gradually, as the machine developed, the distinction between the kinds of tapes became blurred. As matters stand now, most orders are contained in the routine tapes, but some may be put into the problem tape; some numerical information may be put into routine tapes, and most of it is distributed between the table and problem tapes at will, to fit the needs of a particular problem. Certain information remains restricted to the problem tape, especially that which is to be printed as the heading of the answer sheet; and even this last remaining limitation has proved to be a drawback rather than a useful distinction between tapes.

As a result of the experience gained in the operation of this machine, it appears that the problem tape is an unnecessary complication of the machine, and that its function had better be distributed between the other two types of tapes. The distinction between order tapes and numerical tapes, on the other hand, appears useful. Most of the projected future machines contemplate the use of a single input tape only, and this seems to entail the considerable drawback that an entire new tape has to be made up for each problem to be solved. A division of the input as indicated makes it possible to combine a single order tape with many different numerical tapes, or (less frequently) one numerical tape with various order tapes, and thus to work a large number of problems without having to produce a large amount of new tape. Whether or not it is desirable to have several numerical input tapes, as on our machine, is partly a question of the price paid for them. They can be used to good advantage by putting the numerical information in groups that

are likely to be changed or left undisturbed simultaneously, so that groups of problems can be handled by changing only a single short tape without disturbing the others. For instance, in solving differential equations, one tape may contain the length of an integration step, the desired accuracy and similar constants referring to the numerical process selected; another tape, the initial conditions; a third tape, certain parameters appearing in the equations; etc. As a rule, in going from one problem to the next only one group of numbers will be changed at a time, so that the changeover can be effected with a minimum amount of new tape. On our machine, moreover, the distribution of the numerical input among several tapes has the great advantage of reducing the time required to find any desired place in the tapes. This is of primary importance in this relatively slow machine, but will be a lesser consideration in future electronic machines.

e. Arrangement of Routine Tapes.—The routine tapes are usually made into closed loops. This arrangement was made because most computations, especially large-scale ones, involve repetitive elements. The transmitters into which the routine tapes are placed can move the tapes in one direction only. Occasionally open-end routine tapes are used in cases in which the course of a computation is straightforward without ever returning to an earlier sequence of orders.

Each routine tape is divided into sections. Six sections are provided for, and any larger number can be obtained by means of a relatively simple programming device. As a rule, the machine reads orders consecutively as they appear on the tape, carrying out each order as it is read. It is possible, however, to give an order which will cause the tape transmitter to "hunt" for the beginning of any desired section of the same routine tape, skipping over all intervening orders. Other orders shift the control from one routine tape to another. While hunting, the tape moves about four times as fast as when reading orders.

f. Arrangement of Table Tapes.—The table tapes of this machine are open-end tapes, and the transmitters into which they are fed can move the tapes in either direction. Each table tape is divided into sections called "pages," and each page into sub-sections called "blocks." Each block can contain an unlimited number of numbers. The use of pages is optional, but the use of blocks is mandatory. In order to find a certain number on a table tape, we have to know the page, if any, the block in which it is located, and its place within the block. Pages and blocks are characterized by four-digit numbers and arranged in ascending order of these numbers. For instance, to find the first number in block 1234 of page 5678 of Table Tape B, we give an order to the machine to move Table Tape B to the beginning of page 5678. Next, we order it to move the same tape to the beginning of block 1234. When this block has been found, the machine will at once read the first number in the block and store it in a special storing register called the "table register." The number can now be used by the machine just as any other number in a storing register. As soon as the machine receives an order to clear the table register, it will automatically read the next number on the table tape and put it into the table register. Thus, numbers are taken off the table tape in the order in which they appear in a block. (It is, of course, possible to read off numbers and clear them without using them and subsequently to call in the same block of the table tape and read off the numbers

previously cleared. Thus, at the cost of some time, numbers may be taken off in any order.)

Since the table tapes can move in either direction, blocks or pages may be called for in any desired order. However, the time required to find a given place on the tape is usually long and is one of the major limitations of this machine. As an indication of the order of magnitude involved, it may be stated that each number covers about one inch of tape (ranging from 1.1 inch for a seven-digit number to 0.5 inch for a one-digit number) and that the "hunting" speed is two inches per second. Thus, if a tape is to contain 2000 numbers, the average hunting distance between two points selected at random is 1000 inches and the hunt requires about 8 minutes. If many such random hunts are needed in a problem, the time requirement becomes prohibitive. However, it is usually possible to arrange the input numbers on the table tapes in approximately the order in which they are needed, and such an arrangement reduces the hunting time materially. It should also be mentioned in this connection that the hunting can be overlapped by other machine operations.

Instead of characterizing a block of data on a table tape by a single block number, it is possible to characterize it by two block numbers which bracket in all block numbers to which the data apply. This feature is used especially when the block numbers represent the arguments of functions stored in the table tape. For instance, suppose that we wish to store on a table the function $\sqrt[3]{x}$. For each value of x we perforate on the table tape a block number equal to x (or a linear function of x, with some suitable arrangement for the decimal point), the value of $\sqrt[3]{x}$ and interpolation coefficients, say, up to the third order. If the machine, in the course of a computation, has obtained the value of x accurate to seven places, it can then be instructed to find block x (to four places) on the table tape, read out the value of the function and of the interpolation coefficients, and by means of interpolation find the exact value of the function for the given x. It may be desired to perforate the value of function and interpolation coefficients, not for every value of x but only for a few selected values, and to take care of the interval between these by interpolation. Thus, we may use one block for all values of x between zero and 0100, another block from 0101 to 0200, etc. Such intervals are called "hyphenated blocks." They may vary in length. For instance, we may continue with blocks 0201–0400, 0401–0700, 0701–1200, etc. Evidently it is desirable to use short blocks in regions in which the function or one of its derivatives changes abruptly, as is the case for $\sqrt[3]{x}$ in the neighborhood of the origin. Also, hyphenated block numbers may be interspersed with simple ("common") ones.

When the machine receives an order to find a given block number x on a certain table tape, it starts to move the tape forward (i.e., in the direction of increasing block numbers) until it encounters a block number. If this is a common block number, it will then move the tape forward if the block number is smaller than x, move the tape backward if the block number is larger than x, and in case it is exactly equal to x, it will read the number following the block number into the table register. If, on the other hand, the machine encounters a "hyphenated block number," it will move the tape forward or backward if x is outside the interval indicated by the block

number, and read the next number into the table register if x is inside that interval (boundaries included).

The storage tape, mentioned in Section 4c above, has the same arrangement as a table tape. This tape runs through a reperforator, from there into a bin, and out of it into a tape transmitter. On orders from the machine the reperforator puts page or block numbers onto the tape, as well as the intermediate results which are to be stored. (The orders must be such that page and block numbers will be perforated in ascending order.) The storage tape transmitter can subsequently be ordered to find a given location (page and block) on the storage tape, and it can move the tape in either direction to find that location. A special safeguarding device ensures that the tape will not be torn if the transmitter hunts for a block which is located close to the reperforator, or which, by mistake, has not yet been perforated on the tape. If the storage tape is used only for final results, it is not necessary to put page or block numbers on it.

We mentioned before that the table tapes of this machine are open-end tapes of arbitrary length in which the block numbers are arranged in order of magnitude but otherwise arbitrary. In contrast, the Harvard machines use looped tapes in which the arguments must be equidistant. The latter arrangement enables the machine to find any desired location much faster, because the distance over which the tape must be moved is determined by the value of the argument alone and not by the contents of the tape. On the other hand, this arrangement has the drawback that the machine operator is limited in the way in which he organizes the numerical input.

g. Arrangement of the Problem Tape.—The problem tape is an open-end tape which can move in one direction only. Among its functions are to signal the beginning and the end of a problem or of certain subdivisions of a problem, to provide a check on the correct loading of the routine and table tape associated with the given problem, to provide for the printing of headings on the answer sheet. Additional functions which may or may not be assigned to it are to give certain orders which are needed only once during a problem (usually at the start), and to supply part or all of the numerical input for the problem.

The problem tape may be divided into sections called "PCS" sections, and these may be subdivided into minor sections called "CCS" sections. These designations are not particularly suggestive; they are abbreviations for "problem cycle section(s)" and "computing cycle section(s)." A "PCS" may contain several "CCS" and in addition two special kinds of subsections, called "SWR" (switch to routine) and "SWP" (switch to printer) sections. The former contains orders similar to those normally stored on routine tapes, the latter contains instructions to the printer, usually concerning the printing of headings. The "CCS" contain numbers; they are similar to table tapes except that they are not subdivided into pages and blocks. Such a subdivision would be pointless, since the problem tape can move only in one direction and therefore all information has to be taken off in the order in which it is used.

The problem tape exerts supreme control over the progress of a problem. When the machine is started the problem tape is the first to move. Typically, it will contain a few orders which direct the machine to make sure that the proper routine and table tapes have been loaded into the proper transmitters.

It may give additional orders (as indicated above) at this time. It will then direct the printer to print the desired headings. There may then follow the first CCS section. Whenever the problem tape comes to the beginning of a CCS, it delegates the control of the problem to the first routine tape. This tape now begins to move, and the machine reads from it the orders pertaining to the computation. In the course of the computation it may draw numbers from the CCS of the problem tape (in the order in which they appear there) and from the table tapes (in any desired order). The control may pass to other routine tapes. Finally it will come to an order that restores the supreme control of the problem tape. When this order is received, the routine tapes cease to move, the machine checks to see that it has received from the problem tape an indication that the present CCS section is finished, and then moves the problem tape to the beginning of the next section. If this is again a CCS, control is at once delegated to the routine tapes; if it is a SWR or SWP section, control rests with the problem tape for the length of the section. This process continues until the machine arrives at the end of the problem tape.

(To be concluded)

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[1] *MTAC*, v. 2, p. 185f.
[2] *MTAC*, v. 2, p. 97f, and 366.

The Square Root Method for Solving Simultaneous Linear Equations

The square root method for solving a system of linear equations was probably first developed by BANACHIEWICZ[1] in 1938, but it was independently developed by DWYER.[2] The growing popularity of the method in the U. S. stems from the papers by Dwyer and the simple explanation of the method given by DUNCAN & KENNEY.[3] The method is directly applicable to solving a system of linear equations only when the matrix of the coefficients of the unknowns is symmetric. However, if this is not the case, the system represented symbolically by the matrix equation $MX = N$, where M is the matrix of the coefficients of the unknowns, X is the column matrix of the unknowns, and N is the column matrix of the right hand members of the equations, can be transformed into another matrix equation with a symmetric matrix as the coefficient of X. This is accomplished by premultiplying both sides of the equation by the inverse of M. The additional computations required to symmetrize the matrix M, before applying the square root method, makes it doubtful that this method is as efficient as other more direct methods when the coefficients of the unknowns in the original equations do not form a symmetric matrix.

The NBSCL has applied the square root method for solving systems of normal equations arising in least square solutions and has found the method very efficient. With a standard calculating machine a good computer can solve ten equations in ten unknowns carrying about eight significant digits

(actually a fixed number of decimal places is carried) in less than three hours. The method is equally efficient for solving any system of linear equations with symmetric coefficients in spite of the fact that some of the computations may involve imaginary quantities, as will be seen later.

For a brief description of the method, consider the system

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = g_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = g_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = g_n \end{array}$$

where the matrix of the coefficients, A, is assumed to be symmetric. This system can be represented by the matrix equation

$$AX = G \tag{1}$$

where X and G are the column matrices $[x_i]$ and $[g_i]$ respectively. The matrix equation (1) can be transformed into an easily solvable triangular matrix equation

$$SX = K \tag{2}$$

by defining the triangular matrix S by

$$S'S = A \tag{3}$$

where S is of the form

$$S = \begin{pmatrix} s_{11} s_{12} \cdots s_{1n} \\ 0 \;\; s_{22} \cdots s_{2n} \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ 0 \cdots\cdots 0, \; s_{nn} \end{pmatrix}$$

and S' is the inverse of S.

Substituting $S'S$ for A in (1), we get

$$S'SX = G.$$

Therefore if K is a column matrix, such that

$$S'K = G, \tag{4}$$

it follows that $SX = K$.

Once the S and K matrices are known, the X matrix can be found very easily from the last equation.

Thus there are three steps required in the square root method.

Step I—To obtain S from $S'S = A$

By equating elements of the product $S'S$ with corresponding elements of A, we get $\sum_{r=1}^{i} s_{ri}s_{rj} = a_{ij}(j \geqslant i)$; therefore $s_{11} = \sqrt{a_{11}}$, $s_{1j} = a_{1j}/s_{11}$, and $s_{ii} = (a_{ii} - \sum_{r=1}^{i-1} s_{ri}^2)^{\frac{1}{2}}(i > 1)$, $s_{ij} = (a_{ij} - \sum_{r=1}^{i-1} s_{ri}s_{rj})/s_{ii}\ (j > i)$.

Step II—To obtain K from $S'K = G$

By equating corresponding elements of $S'K$ and G, we get $\sum_{r=1}^{i} s_{ri}k_r = g_i$; therefore $k_1 = g_1/s_{11}$, and $k_i = (g_i - \sum_{r=1}^{i-1} s_{ri}k_r)/s_{ii}\ (i > 1)$.

Step III—To obtain X from $SX = K$

By equating corresponding elements of SX and K, we get $\sum_{r=i}^{n} s_{ir}x_r = k_i$; therefore $x_n = k_n/s_{nn}$, and $x_i = (k_i - \sum_{r=i+1}^{n} s_{ir}x_r)/s_{ii}\ (i < n)$.

To afford a check on the numerical work, a second system of n linear equations obtained by replacing x_i by $\bar{x}_i - 1$ is also solved. This second matrix equation can be written as

$$A\bar{X} = \bar{G}$$

where the elements of $\bar{G} = [\bar{g}_i]$ are given by $\bar{g}_i = \sum_{r=1}^{n} a_{ir} + g_i$, which is the sum of all the coefficients plus the constant term in the ith row. Since A is the same as in the original matrix equation, only $\bar{K}$ and $\bar{X}$ need be computed to solve this second system. $\bar{K}$ is obtained from $S'\bar{K} = \bar{G}$ in the same way as K is obtained in Step II above, the only difference being that g_i is replaced by $\bar{g}_i$ and k_r by $\bar{k}_r$. $\bar{X}$ is then obtained from $S\bar{X} = \bar{K}$ as shown in Step III except that x_r is replaced by $\bar{x}_r$ and k_i by $\bar{k}_i$.

The computations are checked by means of the following equalities:

$$\bar{k}_i = k_i + \sum_{r=i}^{n} s_{ir} \tag{5}$$

$$\bar{x}_i = x_i + 1 \tag{6}$$

which should be satisfied except for rounding errors.

For the checking procedure to be efficient, the computations should proceed in the following order:

(a) Obtain all the elements of $\bar{G}$.

(b) Obtain the entire first row of S, K and $\bar{K}$, and check $\bar{k}_1$ by (5), then obtain the entire second row and check $\bar{k}_2$ by (5) etc., through the nth row.

(c) Obtain x_n and $\bar{x}_n$ and check by (6), then obtain x_{n-1} and $\bar{x}_{n-1}$ and check by (6), and so on until x_1 and $\bar{x}_1$ are obtained and checked.

In this manner, each row is checked before proceeding to the next row, and each x is checked before proceeding to the next x.

In applying the square root method, all computations should be performed on a calculating machine writing down only the final results in the following compact and systematic arrangement:

$$
\begin{array}{ccccc|c|c}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & g_1 & \overline{g_1} \\
 & a_{22} & a_{23} & \cdots & a_{2n} & g_2 & \overline{g_2} \\
 & & a_{33} & \cdots & a_{3n} & g_3 & \overline{g_3} \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & a_{nn} & g_n & \overline{g_n} \\
\hline
s_{11} & s_{12} & s_{13} & \cdots & s_{1n} & k_1 & \overline{k_1} \\
 & s_{22} & s_{23} & \cdots & s_{2n} & k_2 & \overline{k_2} \\
 & & s_{33} & \cdots & s_{3n} & k_3 & \overline{k_3} \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & s_{nn} & k_n & \overline{k_n} \\
\hline
x_1 & x_2 & x_3 & \cdots & x_n & & \\
\overline{x_1} & \overline{x_2} & \overline{x_3} & \cdots & \overline{x_n} & & \\
\end{array}
$$

It should be noted that the diagonal elements of the S matrix involve square roots which may lead to imaginary elements. This would never occur with normal equations but could arise when A is an arbitrary symmetric matrix. However, the occurrence of pure imaginary elements in S causes no difficulty in applying the method.

JACK LADERMAN

NBSCL

[1] TADEUSZ BANACHIEWICZ, (a) "Principes d'une nouvelle technique de la méthode des moindres carrés"; (b) "Méthode de résolution numérique des équations linéaires, du calcul des déterminants et des inverses, et de réduction des formes quadratiques," Akademija Umiejetnosci, Krakow, *Wydział Matematyczno-przyrodniczy, Bull. Intern.*, s.A., *Sci. Math.*, 1938, p. 134–135; 393–404.

[2] P. S. DWYER, "A matrix presentation of least squares and correlation theory with matrix justification of improved methods of solution," *Annals Math. Stat.*, v. 15, 1944, p. 82–89; and "The square root method and its use in correlation and regression," Amer. Stat. Assn., *Jn.*, v. 40, 1945, p. 493–503.

[3] DAVID B. DUNCAN and JOHN F. KENNEY, *On the Solution of Normal Equations and Related Topics*. Ann Arbor, Edwards Bros., 1946, 35 p.

Coefficients for Expressing the First Twenty-Four Powers in Terms of the Legendre Polynomials

The following table of coefficients gives the exact expression for x^n, $n = 0(1)24$, in terms of Legendre polynomials $P_m(x)$ (written simply as P_m). This extends the previous short tables (inadequate for many needs) which are given in T. M. MACROBERT, *Spherical Harmonics*, London, 1927, p. 96, $n = 0(1)10$; in W. E. BYERLY, *Elementary Treatise on Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics*. Boston, 1895, p. 179, $n = 0(1)8$; and in E. W. HOBSON, *Theory of Spherical and Ellipsoidal*

Harmonics. Cambridge, 1931, p. 45, $n = 0(1)7$. The coefficients were calculated from the following formulae:
For n even:

$$x^n = \frac{1}{(n+1)} P_0 + 5\frac{n}{(n+1)(n+3)} P_2 + 9\frac{n(n-2)}{(n+1)(n+3)(n+5)} P_4 + \cdots + (2n+1)\frac{n(n-2)\cdots 2}{(n+1)(n+3)\cdots(2n+1)} P_n.$$

For n odd:

$$x^n = \frac{3}{(n+2)} P_1 + 7\frac{(n-1)}{(n+2)(n+4)} P_3 + 11\frac{(n-1)(n-3)}{(n+2)(n+4)(n+6)} P_5 + \cdots + (2n+1)\frac{(n-1)(n-3)\cdots 2}{(n+2)(n+4)\cdots(2n+1)} P_n.$$

As a check, all the coefficients of P_m (except those of P_0 or P_1) were recalculated by a different method from the expression

$$(2m+1)\frac{n(n-1)\cdots(n-m+2)}{(n+m+1)(n+m-1)\cdots(n-m+3)}.$$

An additional overall check, for each n, was provided by taking the sum of the coefficients of P_m, which is equal to 1.

These coefficients are useful in obtaining an approximation for polynomials of high degree, which is best in a well-known least square sense. When any function in the interval $[a, b]$ (which is first transformed into $[-1, 1]$) is expanded in terms of Legendre polynomials, (denoting the normalized function by $f(x)$ and the partial sum of degree r by $q_r(x)$), then

$$\int_{-1}^{1} |f(x) - q_r(x)|^2 dx < \int_{-1}^{1} |f(x) - r_r(x)|^2 dx,$$

where $r_r(x)$ is any polynomial of degree $\leqslant r$ that differs from $q_r(x)$. For methods of using these coefficients, one might consult the very instructive article by C. LANCZOS, "Trigonometric interpolation of empirical and analytic functions," *J. Math. Physics*, v. 17, 1938, p. 123–199. Lanczos deals mainly with "economization" using Chebyshev polynomials, but indicates (p. 143–145) that his same methods are applicable to Legendre polynomials, if one desires approximation in this least square sense.

$$\begin{aligned}
1 &= P_0 \\
x &= P_1 \\
3x^2 &= P_0 + 2P_2 \\
5x^3 &= 3P_1 + 2P_3 \\
35x^4 &= 7P_0 + 20P_2 + 8P_4 \\
63x^5 &= 27P_1 + 28P_3 + 8P_5 \\
231x^6 &= 33P_0 + 110P_2 + 72P_4 + 16P_6 \\
429x^7 &= 143P_1 + 182P_3 + 88P_5 + 16P_7 \\
6435x^8 &= 715P_0 + 2600P_2 + 2160P_4 + 832P_6 + 128P_8 \\
12155x^9 &= 3315P_1 + 4760P_3 + 2992P_5 + 960P_7 + 128P_9 \\
46189x^{10} &= 4199P_0 + 16150P_2 + 15504P_4 + 7904P_6 + 2176P_8 + 256P_{10} \\
88179x^{11} &= 20349P_1 + 31654P_3 + 23408P_5 + 10080P_7 + 2432P_9 + 256P_{11}
\end{aligned}$$

$$
\begin{aligned}
6\,76039x^{12} &= 52003P_0 + 2\,08012P_2 + 2\,20248P_4 + 1\,33952P_6 + 50048P_8 \\
&\quad + 10752P_{10} + 1024P_{12} \\
13\,00075x^{13} &= 2\,60015P_1 + 4\,28260P_3 + 3\,54200P_5 + 1\,84000P_7 + 60800P_9 \\
&\quad + 11776P_{11} + 1024P_{13} \\
50\,14575x^{14} &= 3\,34305P_0 + 13\,76550P_2 + 15\,64920P_4 + 10\,76400P_6 + 4\,89600P_8 \\
&\quad + 1\,45152P_{10} + 25600P_{12} + 2048P_{14} \\
96\,94845x^{15} &= 17\,10855P_1 + 29\,41470P_3 + 26\,41320P_5 + 15\,66000P_7 + 6\,34752P_9 \\
&\quad + 1\,70752P_{11} + 27648P_{13} + 2048P_{15} \\
3005\,40195x^{16} &= 176\,78835P_0 + 744\,37200P_2 + 893\,24640P_4 + 673\,17120P_6 \\
&\quad + 352\,12032P_8 + 128\,88064P_{10} + 31\,74400P_{12} + 4\,75136P_{14} \\
&\quad + 32768P_{16} \\
5834\,01555x^{17} &= 921\,16035P_1 + 1637\,61840P_3 + 1566\,41760P_5 + 1025\,29152P_7 \\
&\quad + 481\,00096P_9 + 160\,62464P_{11} + 36\,49536P_{13} + 5\,07904P_{15} \\
&\quad + 32768P_{17} \\
22687\,83825x^{18} &= 1194\,09675P_0 + 5117\,55750P_2 + 6408\,07200P_4 + 5183\,41824P_6 \\
&\quad + 3012\,58496P_8 + 1283\,25120P_{10} + 394\,24000P_{12} + 83\,14880P_{14} \\
&\quad + 10\,81344P_{16} + 65536P_{18} \\
44181\,57975x^{19} &= 6311\,65425P_1 + 11525\,62950P_3 + 11591\,49024P_5 + 8196\,00320P_7 \\
&\quad + 4295\,83616P_9 + 1677\,49120P_{11} + 477\,38880P_{13} + 93\,96224P_{15} \\
&\quad + 11\,46880P_{17} + 65536P_{19} \\
3\,44616\,32205x^{20} &= 16410\,30105P_0 + 71349\,13500P_2 + 92468\,47896P_4 + 79149\,97376P_6 \\
&\quad + 49967\,35744P_8 + 23893\,30944P_{10} + 8619\,52000P_{12} \\
&\quad + 2285\,40416P_{14} + 421\,72416P_{16} + 48\,49664P_{18} + 2\,62144P_{20} \\
6\,72822\,34305x^{21} &= 87759\,43605P_1 + 1\,63817\,61396P_3 + 1\,71618\,45272P_5 \\
&\quad + 1\,29117\,33120P_7 + 73860\,66688P_9 + 32512\,82944P_{11} \\
&\quad + 10904\,92416P_{13} + 2707\,12832P_{15} + 470\,22080P_{17} \\
&\quad + 51\,11808P_{19} + 2\,62144P_{21} \\
26\,30123\,70465x^{22} &= 1\,14353\,20455P_0 + 5\,03154\,10002P_2 + 6\,70872\,13336P_4 \\
&\quad + 6\,01471\,56784P_6 + 4\,05955\,99616P_8 + 2\,12746\,99264P_{10} \\
&\quad + 86835\,50720P_{12} + 27224\,10496P_{14} + 6354\,69824P_{16} \\
&\quad + 1042\,67776P_{18} + 107\,47904P_{20} + 5\,24288P_{22} \\
51\,45894\,20475x^{23} &= 6\,17507\,30457P_1 + 11\,74026\,23338P_3 + 12\,72343\,70120P_5 \\
&\quad + 10\,07427\,56400P_7 + 6\,18702\,98880P_9 + 2\,99582\,49984P_{11} \\
&\quad + 1\,14059\,61216P_{13} + 33578\,80320P_{15} + 7397\,37600P_{17} \\
&\quad + 1150\,15680P_{19} + 112\,72192P_{21} + 5\,24288P_{23} \\
806\,19009\,20775x^{24} &= 32\,24760\,36831P_0 + 143\,32268\,30360P_2 + 195\,70959\,47664P_4 \\
&\quad + 182\,38170\,12160P_6 + 130\,09044\,42240P_8 + 73\,46283\,90912P_{10} \\
&\quad + 33\,09136\,89600P_{12} + 11\,81107\,32288P_{14} + 3\,27809\,43360P_{16} \\
&\quad + 68380\,26240P_{18} + 10103\,02976P_{20} + 943\,71840P_{22} + 41\,94304P_{24}
\end{aligned}
$$

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A New Approximation to π

(conclusion)

In *MTAC*, v. 2, p. 320, it was announced that the values of π and $\tan^{-1} \frac{1}{8}$, in our previous joint article with Mr. Smith, p. 245–248, called for correction beyond 722D. We are now in a position to guarantee the accuracy of our 808D values of π, $\tan^{-1} \frac{1}{5}$, $\tan^{-1} \frac{1}{239}$, $\tan^{-1} \frac{1}{4}$, $\tan^{-1} \frac{1}{20}$, and $\tan^{-1} \frac{1}{1985}$.

This certainty was achieved by Mr. Ferguson carrying through his calculations to 812D, with the independent formula which he had been using. In this way he discovered certain errors in the work of Dr. Wrench.

Dr. Wrench independently discovered another error in his work. Thus 12 digits have to be changed in the previously published value of π in the interval 721D–808D. The correct sequence is as follows:

$\pi =$	.....	.....	.....	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	594(55)								

The corrected sequence in $\tan^{-1} \frac{1}{5}$, 721D–808D is as follows:

.....	.....	.....	.....	40468	13622	41107	68081	00362	13759
92595	89071	66648	51452	55706	79954	88100	43132	95466	83892
29036	883(09)								

There is no change to be made in Mr. Smith's previously published values of $\tan^{-1} \frac{1}{239}$ to 811D or in Mr. Ferguson's (a) $\tan^{-1} \frac{1}{4}$, (b) $\tan^{-1} \frac{1}{20}$, (c) $\tan^{-1} \frac{1}{1985}$, each to 710D. We add, however, Mr. Ferguson's values of (a), (b), and (c) 711D–808D.

(a)	.....	.....	59314	07269	83047	72505	80573	53263	40402	52936
	68088	15266	48595	21663	12393	77666	38170	19070	41862	22868
	53718	412(24)								
(b)	.....	.....	66995	11975	27660	30938	09302	51266	37651	02972
	47760	03233	34506	66330	41815	96327	61997	38784	39560	96639
	31126	113(14)								
(c)	.....	.....	35385	21705	44797	37589	88930	29687	78069	65707
	60943	18995	57207	68639	53447	83160	99985	33336	38876	42719
	95279	798(75)								

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RECENT MATHEMATICAL TABLES

450[A, B, C, D].—DONALD V. MITCHELL, (a) *Six-Place Tables for Precision Computing to accompany* [b] *Streamlined Methods of Computing with Slide Rule and Mathematical Tables* (a) 1945, 47 p., stiff paper cover $1.00; [b] revised and greatly enlarged ed., 1947, 80 p., stiff paper cover $1.00, Seattle, Washington, Craftsman Press. 13.5 × 20.5 cm. Procurable from the author, 12345 Sand Point Way, Seattle 55, Wash.

The tables are: Log N, $N = 1000$–9999, with Δ; the six natural trigonometric functions and their logarithms for each tenth of a degree, with c.d.; decimal equivalents of common fractions; N^2, N^3, $N^{\frac{1}{2}}$[4D], $N^{\frac{1}{3}}$[4D], $N = 1(1)1000$. The "Slide Rule Topics" in [b] occupy p. 37–58.

451[A, D].—FRIEDRICH SCHULZE, *Hilfstafeln für die Schrägmessung mit 5 m-Latten und mit dem 20 m. und dem 10 m-Stahlband.* Liebenwerda, Verlag von R. Reiss, 1941 (?) 23 p. 14 × 16 cm. Limp cover. 50 pfennige.

T. 1, arctan $(h/5)$, $h = .1(.01)2.09$, to the nearest 0′.1.
T. 2–3, $Z = (25 - h^2)^{\frac{1}{2}} - 5000$, and $0.2Z$, $h = .1(.01)2.09$, h in m. and Z in mm.
T. 4–5, $r = 5000 - 25/(25 + h^2)^{\frac{1}{2}}$ and $0.2r$, $h = .1(.01)2.09$.
T. 6–7, $Z = 20(\sec\alpha - 1)$ and $0.05Z$, $\alpha = 0(0°.1)24°.9$.
T. 8–9, $r = 20(1 - \cos\alpha)$ and $0.05r$, $\alpha = 0(0°.1)24°.9$.

T. 10–11, $Z = 10(\sec \alpha - 1)$ and $0.1Z$, $\alpha = [24°(0°.1)48°.9$; mostly 3S].
T. 12–13, $r = 10(1 - \cos \alpha)$ and $0.1r$, $\alpha = [24°(0°.1)48°.9$; mostly 3S].
T. 14, $r = Z - Z^2/(5000 + Z)$, Z in mm. = 0(2)408.
T. 15, $r = Z - Z^2/(2000 + Z)$, Z in cm. = 0(1)209.
T. 16, $r = Z - Z^2/(1000 + Z)$, Z in cm. = 80(1)349.

Errata: T. 1, $h = 0.42$, *for* 49.0, *read* 48.1; T. 10, $\alpha = 48°.7$, *for* 575, *read* 515.

452[C, D].—OTTO MÜLLER & MICHELE RAJNA, *Tavole di Logaritmi con Cinque Decimali. Ventinovesima edizione riveduta per cura di* LUIGI GABBA. (*Manuali Hoepli*). Milan, Hoepli, 1940. xxxii, ii, 203 p. 10.2 × 15 cm.

The tables in this volume are as follows: **T. 1**, p. 1–33, log N, $N = 1(1)9999$ with Δ. From 1000 to 9990 the corresponding number of degrees, minutes and seconds are given, 16′40″ to 2°46′30″. Values of S and T are also given 1000″(100″)9900″. **T. 2**, p. 35–125, log sin, log tan, log cot, log cos at interval 1′ with Δ and P. P. **T. 3**, p. 139–187, Addition and Subtraction logarithms, $A, -D = 0(.001)2(.01)4(.1)5$, $I\Delta$, $fPPd$; $S, -D = 0(.00001).051(.0001).4(.001)1.8(.01)4(.1)5$, PPd. On p. 33–34 are various constants, mainly functions of π and M; on p. 126–127, lengths of arc of unit circle for each degree, minute, and second; on p. 128 dimensions of the earth; on p. 129–137, 4D tables of the natural trigonometric functions sin, tan, cot, cos, at interval 10′; on p.138, value of gravity at different latitudes and elevations; on p. 188–191, conversion tables of degrees, minutes and seconds to time; on p. 192–194, of degrees and minutes to seconds; on p. 195, physical constants; and on p. 196–203, N^2, $N = 0(1)1000$.

This work was originally compiled by Müller, and first published by Hoepli at Milan in 1883; second ed. 1886, 143 p.; third ed. 1891, 142 p. The table of addition and subtraction logarithms was added "per cura Michele Rajna" (1854–1920), astronomer at the Royal Observatory in Milan, in the fourth ed. 1895 (xxxvi, 185 p.). The fifth ed. 1897, sixth ed., 1900 (xxxvi, 191 p.), seventh ed. 1903, eighth ed. 1905, ninth ed. 1906, tenth ed. 1908, eleventh ed. 1911, and the twelfth and thirteenth eds., 1915, and the fifteenth ed., 1922, were issued in similar form, and with the same number of pages. But with the twentieth revised ed., in 1926, "per cura LUIGI GABBA," also astronomer at the Milan Observatory, Müller & Rajna are listed as joint authors. In this edition, the twenty-fifth, 1932, the twenty-sixth, 1935, and in the present twenty-ninth ed., the number of pages is the same except for the leaf of 17 errata in **T. 2**, added in the last edition; but the first three of these were corrected. Four of these errors (p. 106, 114, 124, 125) were also in the twenty-fifth edition, but none of those four occurred in the fifteenth edition.

The following errata of the present volume were noted: p. 63, cos 13°55′, *for* 9.98906, *read* 9.98706; p. 100, cot 32°5′, *for* 0.20381, *read* 0.20281; cot 32°6′, *for* 0.20203, *read* 0.20253; cot 32°20′, *for* 0.19960, *read* 0.19860; p. 105, cos 34°54′, *for* 9.91386, *read* 9.91389; p. 192, col. 4, last l., *for* 210 000, *read* 216 000.

R. C. A. & S. A. J.

EDITORIAL NOTE: In the preparation of this review we were greatly assisted by H. E. MOSE, reference librarian of the John Crerar Library, Chicago. Müller is the author of two other tables, namely: *Tavole per la Determinazione del Tempo dietro le Altezze del Sole o di una Stella*. Milan, Hoepli, 1881, xxi, 34 p.; and *Hilfstafeln für praktische Messkunde nebst logarithmisch-trigonometrischen Tafeln*. Zürich, F. Schulthess, 1897, 144 p.

453[D].—ANT. PROKEŠ, *Tabured. Tabellen zur Reduktion von schräg gemessenen Entfernungen für zentesimale Kreisteilung*. Berlin-Grunewald, Wichmann, 1943, 53 p. 16.8 × 24.1 cm.

Let O be a point of observation of an object P whose projection on the horizontal plane is P'. If $OP = m$, $OP' = D$, angle $POP' = \alpha^g$, $D = m \cos \alpha = m - r$, where $r = m(1 - \cos \alpha)$. The table is of r, that is, multiples of versed sin α, for $m = 10(10)100$, $\alpha = [0(2^c)50^g$; 3D], Δ. Random checking revealed an error in r, $\alpha = 49^g.26$, *for* 28.372, *read* 28.472. While there

are numerous tables of versed sine for argument in degrees, minutes and seconds, and also tables with argument in radians and time, we recall only one other table with centesimal argument, namely the Chinese table by E. A. SLOSSE, to which we referred, *MTAC*, v. 1, p. 38: *Tables des Valeurs Naturelles des Expressions Trigonométriques. Division Centésimale* . . . 1923.

R. C. A.

454[D, Q].—D. K. KULIKOV, "Novyĭ metod obrabotki nabli͡udeniĭ par Tsingera" [New method for the treatment of observations of Zinger star-pairs], Akad. N., SSSR, Institut Teoreticheskoĭ Astronomii, Bi͡ulleten', v. 4, no. 2(55), March, 1947, p. 77–86. 19.3 × 25.1 cm.

There is a 3D table on p. 82 of $\sigma_0(n) = \frac{1}{8}n^3 \sin^2 1^{\mathrm{s}}$ for $n = 300(10)610(5)800$. There are 24 entries for the first 299^{s}, such as $n = 0 - 82^{\mathrm{s}}$, $\sigma_0(n) = .000$; $n = 82^{\mathrm{s}} - 119^{\mathrm{s}}$, $\sigma_0(n) = .001°$; $n = 119^{\mathrm{s}}$–141^{s}, $\sigma_0(n) = .002°$, $\cdots$; $n = 295^{\mathrm{s}}$–299^{s}, $\sigma_0(n) = .023°$.

455[E, M].—ADRIAAN VAN WIJNGAARDEN, *Enige Toepassingen van Fourier-integralen op Elastische Problemen*. Diss. Delft Technische Hoogeschool. Delft, 1945. 125 p. 15.5 × 24.5 cm.

T. VII. $\int_0^\infty t^p \cosh^n \pi t dt/\sinh^m \pi t$, 31 exact values, $n = 0(1)3$, $m = 1(1)6$, $p = 1(1)6$.

T. VIII. $\sum_{k=N}^{\infty} (2k+1)^{-n}$, $N = 0(1)10$, $n = [2(1)6; 5D]$.

T. IX. $\int_0^\infty t \sinh pt \coth qt dt/\sinh t$, $q = .1(.1)1$, $p = [0(.1).9$; mostly 4D].

T. X. $\sum_{k=N}^{\infty} k^{-n}$, $N = 1(1)10$, $n = [2(1)6; 5D]$.

T. XI. $\int_0^\infty t \sinh^2 pt \coth qt dt/\sinh^2 t$, $q = .1(.1)1.8$, $p = [0(.1).9$; mostly 4D].

T. XII. $\int_0^\infty t^2 \coth rt dt/\sinh t$, $r = [.1(.1)1; 4D]$.

T. XIII. $\int_0^\infty t^3 \coth rt dt/\sinh^2 t$, $r = [.1(.1)1; 4D]$.

456[F].—A. GLODEN, "Factorisation de Nombres de la forme $16x^8 + 1$," *Euclides*, Madrid, v. 7, 1947, p. 95. 16.6 × 24.1 cm.

This note gives 40 factorizations of the function $16x^8 + 1$ for values of x between 23 and 94 inclusive. Seven of these are incomplete, the missing prime factors being in excess of $5 \cdot 10^8$. Twelve of the complete factorizations had been given previously by the author (see *MTAC*, v. 2, p. 300) in a more extensive table of the function $x^4 + 1$. The present tables are based on the inversion of tables of solutions of the quartic congruence $y^4 + 1 \equiv 0 \pmod{p}$, *MTAC*, v. 2, p. 71–72, 210–211, 252, 300.

D. H. L.

457[F].—ALBERT GLODEN, *Liste des formes linéaires des nombres dont le carré se termine dans le système décimal par une tranche donnée de 4 chiffres*. Luxembourg, author, rue Jean Jaurès 11, 1947. 15 leaves mimeographed. 21.1 × 29.9 cm.

This interesting table gives not only all four digit endings of squares but also, for each ending, the sort of numbers whose squares have that particular ending. Tables giving

merely the endings, arranged in increasing sequence, have been given before by KULIK,[1] SCHADY,[2] and THÉBAULT.[3] There are precisely 1044 such endings of four digits. The numbers whose squares have these endings are described by linear forms $mn \pm a (n = 0, \pm 1, \pm 2 \cdots)$. Thus we have the entry

$$1801 \qquad 5000n \pm 349 \qquad 5000n \pm 901$$

indicating that numbers congruent to ± 349 and ± 901 modulo 5000 and no others have squares ending in 1801. Of the 1044 possible endings 500 have $m = 5000$, as above; 500 have $m = 2500$; 20 have $m = 1000$; 20 have $m = 500$; two have $m = 200$; and two have $m = 100$. The table is useful not only in recognizing non-squares by their endings but also in setting up excluding programs by means of quadratic congruences.

D. H. L.

[1] J. P. KULIK, *Tafeln der Quadrat- und Kubik-Zahlen aller natürlichen Zahlen bis Hundert Tausend, nebst ihrer Anwendung auf die Zerlegung grosser Zahlen in ihre Factoren*. Leipzig, 1848.

[2] SCHADY, "Tafeln für die dekadischen Endformen der Quadratzahlen," *Jn. f. d. reine u. angew. Math.*, v. 84, 1878, p. 85–88.

[3] V. THÉBAULT, "Sur les carrés parfaits," *Mathesis*, v. 48, Oct. 1934 Suppl., 22 p. See also *MTAC*, v. 2, p. 72.

458[F].—H. GUPTA, "A table of values of $N_3(t)$," Nat. Inst. Sciences, India, Calcutta, *Proc.*, v. 13, 1947, p. 35–63.

Let $n_2(k)$ and $n_3(k)$ stand for the number of solutions in non-negative integers of the respective equations

$$x^2 + y^2 = k, \qquad x^2 + y^2 + z^2 = k,$$

and let their sums be denoted by

$$N_2(t) = \sum_{k=0}^{t} n_2(k), \qquad N_3(t) = \sum_{k=0}^{t} n_3(k).$$

The main table of this paper gives (p. 39–63) $n_3(t)$ for $t \leq 10^4$. The values of the argument t are classified modulo 8 into seven columns (since $n_3(8m + 7) = 0$ this class of values is omitted). In the right margin the values of $N_3(t)$ for $t = 8m + 7$ are accumulated. Other values of this function may be found quickly by adding or subtracting the appropriate values of $n_3(k)$. This table was built up from a similar manuscript table of the function $n_2(k)$ by means of the formula

$$n_3(t) = \sum_{m=0}^{\infty} n_2(t - m^2).$$

It has been checked by means of the class number relation

$$n_3(8i + 3) = \sum h'[(32i + 12)/d^2],$$

where the sum extends over the square divisors d^2 of $8i + 3$ and where $h'(D)$ denotes the number of properly primitive classes of binary quadratic forms $ax^2 + bxy + cy^2$ of negative determinant $-D = b^2 - 4ac$. The necessary values of h' were taken from the author's table[1] (part I). Conversely, the present table may be used to find isolated values of the class number function.

There is also a small table (p. 37–38) connecting the results of the main table with the number of lattice points inside circles and spheres. Let $C(r)$ and $S(r)$ denote respectively the number of points with integer coordinates inside or on the circle and sphere of radius r (with center at the origin). If points on these loci are given the same weight as interior points then

$$C(r) = 4N_2(r^2) - 4r - 3, \qquad S(r) = 8N_3(r^2) - 12N_2(r^2) + 6r + 5.$$

The table gives for $r = 1(1)100$ the values of

$$N_2(r^2), \qquad C(r), \qquad N_3(r^2), \qquad S(r),$$

and also, for comparison with $S(r)$, the nearest integer V to the volume of the sphere of radius r. At $r = 100$ we find

$$S(100) = 41\,87857, \qquad V(100) = 41\,88790 = 1.000223S(100).$$

D. H. L.

[1] H. GUPTA, "On the class-numbers of binary quadratic forms," Tucuman, Argentina, Universidad, *Revista*, S. A., *Matem. y Fisica Teorica*, v. 3, 1942, p. 283–299. See *MTAC*, v. 1, p. 180–182.

459[F].—D. B. LAHIRI, "On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$.—I," Calcutta Math. Soc., *Bull.* v. 38, Dec. 1946, p. 193–206. 18.7 × 24.2 cm.

In this first part of the paper is found tabular material relating to the function $\sigma_k(n)$ only. This function is the sum of the k-th powers of the divisors of n. The main results of the paper are incorporated in a table giving 89 congruence relations, with respect to various moduli, between the functions $\sigma_k(n)$. One of these, for example, is

$$11\sigma_9(n) \equiv 10(3n-2)\sigma_7(n) + \sigma(n) \pmod{480}.$$

These are derived from tables of relations between the power series

$$\Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n) x^n$$

which are extensions of similar tables of RAMANUJAN.[1] Parts II and III are to contain results on Ramanujan's function $\tau(n)$.

D. H. L.

[1] S. RAMANUJAN, Camb. Phil. Soc., *Trans.*, v. 22, 1916, p. 159–184; *Coll. Papers*, Cambridge, 1927, p. 136–162.

460[F].—KURT MAHLER, "Lattice points in two-dimensional star domains (III)," London Math. Soc., *Proc.*, s. 2, v. 49, p. 168–183, 1946. 17.3 × 25.2 cm.

This paper, presented to the Society in 1942, contains two small tables (p. 178, 181) giving data on lattice points in a domain bounded by two concentric ellipses each of area π. These results are illustrations of the author's general method[1] and are too special to merit a detailed explanation here.

D. H. L.

[1] K. MAHLER, "Lattice points in two-dimensional star domains (I)," London Math. Soc., *Proc.*, s. 2, v. 49, p. 128–157, 1946.

461[F].—HARRY C. ROBERT, JR., "Prime-factor table, numbers $6n \pm 1$, base XII," *Duodecimal Bulletin*, v. 3, no. 2, June 1947, p. (14–17, i.e.) 16–19. 13.7 × 21 cm.

This table gives the factorization into primes, or indicates the primality, of all integers prime to 6 and less than 5184 (= $3 \cdot 12^3$). The arrangement is such that numbers in the same column but in adjacent lines differ by 144. Thus the table has 48 columns headed by the numbers prime to 6 and less than 144. These headings are simply the last two digits of numbers $6n \pm 1$ when written to the base 12. All numbers are written duodecimally.

D. H. L.

462[F].—Arnold Walfisz, "On the additive theory of numbers. X.," Akad. N., SSSR, Gruzinskiĭ Filial, Matem. Inst., *Trudy*, Tiflis, v. 11, 1942, p. 173–186.

This paper dealing with the universality of the two cubic forms $ax_1^3 + x_2^3 + \cdots + x_s^3$, $a = 1, 20$, contains two tables which serve to show that all positive integers n for which $5745 \leq n \leq 11000$ or $12000 \leq n \leq 62000$, except for 5818 and 8042, are sums of not more than 6 positive cubes. Such results for much more extensive ranges have been obtained by von Sterneck[1] and Dickson.[2] The point of the present paper is to show that more extensive tabular evidence is unnecessary.

D. H. L.

[1] R. D. von Sterneck, "Über die kleinste Anzahl Kuben, aus welchen jede Zahl bis 40 000 zusammengesetzt werden kann," Akad. d. Wiss., Vienna, math-natw. Kl., *Sitzb.*, v. 112, section 2a, 1903, p. 1627–1666.

[2] L. E. Dickson, Manuscripts in the University of Chicago Library: (a) Table of the minimum number of cubes required to represent each integer from 40 000 to 270 000; (b) Table of sums of four cubes from 270 000 to 560 000.

463[G, K].—(a) M. Ziaud-Din, "Tables of symmetric functions for statistical purposes," Nat. Acad. Sci., India, *Proc.*, v. 10, 1940, p. 53–60. (b) S. M. Kerawala, "Table of monomial symmetric functions of weight 9," *ibid.*, v. 11, 1941, p. 51–55. (c) S. M. Kerawala and A. R. Hanafi, "The table of symmetric functions of weight 10," *ibid.*, v. 11, p. 56–63. (d) S. M. Kerawala and A. R. Hanafi, "Table of monomial symmetric functions of weight 11," *ibid.*, v. 12, 1942, p. 81–96.

All four tables give the coefficients in the linear representation of the monomial symmetric functions of weight m in terms of products of the sums of like powers. The values of m considered in (a) are 7, 8, and 9. The weights of the other tables are as indicated in their titles. As an example of a relation of weight 7 we find in Table 1 of (a) that

$$2 \sum \alpha_1^5\alpha_2\alpha_3 = S_1^2S_5 - 2S_1S_6 - S_2S_5 + 2S_7. \qquad S_i = \sum \alpha^i.$$

The reason for using the sums S_i as basis instead of the more usual[1] elementary symmetric functions $\sum \alpha_1\alpha_2 \cdots \alpha_i$ is because the S's occur as moments in statistical problems. There is another good reason not connected with statistics: isolated values of the S's are more easily computed than other symmetric functions. The three tables in (a) are so poorly printed that it is difficult to tell which entry lies at the intersection of a given row and column. Also the arrangement of the symmetric functions is not lexicographical so that the tables have an unkempt, jumbled appearance instead of the usual attractive triangular layout. The tables for $m = 1(1)8$ were published in 1938 by Sukhatme.[2] The new table of (a), for $m = 9$, contains a number of errors which are corrected by the republishing of the table in (b). Here, and in (c) and (d), the arrangement and printing are good. There are 5 errors in (c) which are noted in the introduction of (d). The reviewer has checked the table in (d) and finds only one error: In the expansion of

$$720 \sum \alpha_1^3\alpha_2^2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8$$

the coefficient of $S_1^3S_2S_6$ should be -240, not -24.

D. H. L.

[1] See for example F. Faà di Bruno, *Théorie des Formes Binaires*, Turin, 1876, for weights ≤ 11, on the three folding plates; second ed 1883. German ed., Leipzig, 1881.

[2] P. V. Sukhatme, "On bipartitional functions,' R. Soc. London, *Phil. Trans.*, v. 273A, 1939, p. 399.

464[L].—NBSCL, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments.* Second ed., New York, Columbia Univ. Press, 1947. xliv, 403 p. 20 × 26.5 cm. $7.50.

That an edition of 800 copies of this valuable work should have been exhausted in the past four years demonstrated clearly how great was the need for a Table of this kind. For a review of the first edition by Professor Philip M. Morse, see *MTAC*, v. 1, p. 187–189. In the new edition, boosted in price by fifty percent, and in red rather than buff colored jacket, the tabular pages are unchanged, but the incorrect contour charts on p. xv, xvii of the first edition (see *MTAC*, v. 1, p. 326) are now corrected. Several minor changes have been made in the Introduction, p. xxi, xxx–xxxi, and the former Supplementary Note 2, "Expression for differences in terms of derivatives" has been dropped. One title (67) has been added (p. xliv) to the Bibliography (namely, *Guide to Tables of Bessel Functions*, *MTAC*, v. 1, p. 205–308).

When the Bibliography (66 titles, a number of them non-tabular) of the first edition was published in 1943 it served a very useful purpose, and slips and infelicities might have been, to a certain extent, condoned by workers in the field. But when in 1947 the NBS brings out a new edition repeating all the old untidiness, inaccuracies, and gross errors, such careless editing under such high auspices causes no little astonishment.

One may be precise in indicating clearly to what one thus refers. On p. xix and xl, there are incorrect references to Hayashi's, instead of Dinnik's short tables of $J_0(z)$ and $J_1(z)$. The description of the contents of the Badellino entry in (8) is wholly wrong (the correct statement was given in *MTAC*, v. 1, 1944, p. 281). The description of the Smith entry in (58) contains several errors: Michell, not "Mitchell," functions are in question; the definitions of the third and fourth of these functions are incorrect. It is stated in 1947 that items (8) Badellino, (48) Meissel, and (60) "Toelke" (the name is Tölke), had not been seen, yet in 1944 members of the NBSCL staff gave for publication material indicating errors in Tölke (*MTAC*, v. 1, p. 306); since 1944 any alert editor could have in America readily consulted the Badellino and Meissel items. The titles in (6) Anding and (38) Hayashi need repair. In (9) BAASMTC, line 9, the entry should be "$A_i(x)$, $B_i(x)$ used to calculate $J_i(x)$, $Y_i(x)$ to 8D." If the statement in (31) Dwight "with tables reprinted from (14) and (17)" be interpreted as only tables previously published, exactly the same as the statement in (28) Dwight, it is incorrect. In the (45) Mascart & Joubert entry it should have been indicated that the second edition of the work was being considered and the reference to "Kelvin's" table ought to have been to Maclean's (see *MTAC*, v. 1, p. 297). Meissel's table (47) was published in 1889, not 1888. Even the five page numbers xL–xLiv are not corrected.

But more remains to be set forth. Whatever reason may in 1943 have led to the insertion of (64) Brasey, (65) Topping & Ludlam, (66) Bickley & Nayler out of alphabetical order, and 12 entries anonymously under the heading B.A.A.S. when all of the authors are well known (Airey, Alger, Lodge, Savidge, Webster), there was no way in 1947 for the meticulous editor to avoid putting all of these items with the others in one alphabet. He would also naturally in items of his list have noted new editions and given references to important published lists of errata; this would, for example, have included the 1945 edition of Jahnke & Emde, where elaborate corrections were made in at least one Bessel Function table. The insertion in the Bibliography of a selected list of important tables published during 1943–1947, would obviously have rounded out the revision, where of course all possibilities for improvement have not been here suggested.

This will suffice by way of illustration of strictures made above. We have gone into the matter at some length in the hope, that as the NBS brings out new volumes, and later new editions, of its mathematical tables, much greater care may be taken in their preparation so that they may be immediately recognizable as finely edited up-to-date contributions to scholarship. The tables themselves have long been of this character. Our plea is that everything else in such volumes may ever be on an equally high plane of achievement.

R. C. A.

465[L].—NBSCL, *Tables of Spherical Bessel Functions*, volume 2. New York, Columbia University Press, 1947, xx, 328 p., 20 × 26.4 cm. $7.50. V. 1 reviewed by Prof. W. G. BICKLEY, *MTAC*, v. 2, p. 308–309.

The table of $F = (\frac{1}{2}\pi/x)^{\frac{1}{2}}J_\nu(x)$ is continued for $\pm 2\nu = 29(2)43$, for $x = 0(.01)10(.1)25$, and $\pm 2\nu = 45(2)61$, for $x = 10(.1)25$. The entries are given with δ^2 or modified δ^2, and sometimes δ^4, to 8S, 9S, or 10S for $x \leq 10$, and to 7S for $x > 10$, except for a few entries very close to the zeros of the function. For $2|\nu| = 43$ interpolation for F is laborious when $x < 5$ and the interval in x is .01. This difficulty increases with increasing $|\nu|$, so that at $2|\nu| = 61$, very limited accuracy could be obtained by a four-point interpolation formula when $x < 10$. For this reason the functions F have not been tabulated for $x < 10$, $2|\nu| > 43$. However, the closely related functions $\Lambda_\nu(x)$ are tabulated. $F = \pi^{\frac{1}{2}}[2^{\nu+\frac{1}{2}}\Gamma(\nu+1)]^{-1}x^{\nu-\frac{1}{2}}\Lambda_\nu(x)$, or $\Lambda_\nu(x) = 2^\nu\Gamma(\nu+1)J_\nu(x)/x^\nu$. Λ_ν is tabulated, with δ^2 (sometimes modified) and δ^4 (partly), for $x = 0(.1)10$, $2\nu = [1(1)41(2)61; 9D]$, and $x = 10(.1)25$, $2\nu = [1(2)61; 7S$ mostly$]$; also for negative values of ν in regions where F does not difference well, $-2\nu = 29(2)33$, $x = [0(.1)9.5(.05)10(.1)25; 7S$ mostly$]$; $-2\nu = 35(2)61$, $x = [0(.1)25; 7S$ mostly$]$. $\Lambda_n(x)$ is smoother than F and interpolation for $\Lambda_\nu(x)$ in the ν direction is good over a large portion of the $x - \nu$ plane when ν is positive. For these reasons available values of $\Lambda_\nu(x)$ have been included for integral[1] as well as half-integral values of ν, for ν positive and x less than 10. $\Lambda_n(x)$ becomes infinite with $\Gamma(\nu+1)$ for negative integral values of ν; hence interpolation in the ν direction is not feasible when ν is negative and less than -1. For this reason, $\Lambda_n(x)$ is published, for negative ν, only for half-integral values of $\nu < -29/2$, mainly for purposes of interpolation in the x direction in regions where interpolation in the functions F is difficult. There are tables of interpolating coefficients $E_2 = p(1-p^2)/6$, $F_2 = q(1-q^2)/6$, $p + q = 1$, $E_4 = p(1-p^2)(4-p^2)/5!$, $F_4 = E_4(q)$, $G_4 = [p(1-p^2)(4-p^2)/120] - [0.18393p(1-p^2)/6]$, $H_4 = G_4(q)$ at interval .001 in p. Also, tables of $C_\nu = 2^n n!/(2n+1)!$, $2\nu = 2n+1 = [1(2)61; 9D]$; and of $C_\nu = (-1)^n(2n)!/2^n n!$, $2\nu = -2n-1 = [-61(2)-1; 9D]$.

The tables of zeros of $J_\nu(x)$ and of $J_\nu'(x)$ with corresponding values of $J_\nu'(j_{\nu,s})$, and $J_\nu(j'_{\nu,s})$. The values are given 6–10D, for $\pm 2\nu = 1(2)13$, and $s = 1(1)5$ or 6 or 7 or 8; for $\pm 2\nu = 15(2)39$, $s = 1(1)5$ or 4, or 3 or 2, or merely 1. Most of the entries are correct to within a unit in the last tabulated place; those given to at least eleven significant figures are correct to within two units in the last tabulated place.

Extracts from text

[1] Tables of $\Lambda_\nu(x)$ for integral values of ν were published by NBSCL in "Tables of $f_n(x) \cdots$", *Jn. Math. Phys.*, v. 23, 1944, p. 45–60. See *MTAC*, v. 1, p. 363–364.

EDITORIAL NOTE: That the Columbia University Press should have printed the gilt title and volume number of this volume more than an inch higher than the corresponding title and number of volume one is inexcusable.

466[L].—ANDERS REIZ, "On the numerical solution of certain types of integral equations," *Arkiv för Matem., Astr. och Fysik*, v. 29A, no. 29, 1943, 21 p. 13.8 × 21.7 cm.

On pages 6 and 12 are two 7D tables, of zeros (x_i), corresponding CHRISTOFFEL numbers (p_i), and $\alpha_i = \pi^{\frac{1}{2}}p_i e^{x_i^2}$, for T. 1, HERMITE polynomials[1] $H_n(x)$, $n = 2(1)9$; T. 5, LAGUERRE polynomials, $L_n(x)$, $n = 2(1)5$. $H_n(x) = (-1)^n e^{x^2}d^n e^{-x^2}/dx^n$ is here defined as by Hermite, and not as by E. R. SMITH and others (see *MTAC*, v. 1, p. 152–153). In T. 5, the values of zeros of L_5 and the corresponding p_i were previously given by N. S. KOSHLIÂKOV in 1933 (see *MTAC*, v. 1, p. 361, and MTE **121**). For the later much more complete table for zeros of $L_n(x)$ to 8D, and of Christoffel numbers to 8S, $n = 1(1)10$, in SRE/ACS 82, see *MTAC*, v. 2, p. 31. From this table it appears that in Reiz's T. 5, for $n = 5$, the final digits in the second, fourth, and fifth zeros should be respectively 1, 0, 8, instead of 0, 2, 7.

R. C. A.

[1] REIZ refers to A. BERGER, "Sur l'évaluation approchée des intégrales définies simples," K. Vetenskaps Societeten i Upsala, *Nova Acta*, p. 3, v. 16, no. 4, 1893, and states (p. 6) that "Berger has given numerical values for the x's and p's, for $n = 2, 3, 4$," on p. 50 of his paper. Since there are no such values on this page, presumably those given in formulae on p. 52 were meant. So far as I know at present Reiz's table is the first one of $H_n(x)$, in decimal form, which has appeared.

467[L, S, T].—G. W. King, ***Punched-Card Methods in Analyzing Infra-Red Spectra.*** **a.** ***Progress Report,*** April 1–May 31, 1947, issued June, 1947. iii, 44 p. + 3 folding plates. **b.** ***Progress Report,*** June 1–July 31, 1947, issued August, 1947. iii, 14 p. + 2 folding plates. Each 24 × 29 cm. Cambridge, Mass., Arthur D. Little, Inc.

These are two consecutive bimonthly progress reports on work being done under Office of Naval Research contract. The text is produced from typescript and the distribution list is extremely limited, but some of the work (including, it seems and is to be hoped, the main mathematical table) has been submitted to the *Journal of Chemical Physics*. The two reports will be referred to below as the June and August reports. The mathematical theory is contained in the June report, but the August report gives an improved and enlarged version of Table I and an additional Table III.

The work is concerned with the analysis of infra-red band spectra and microwave absorption lines of triatomic molecules of the type of H_2S and its heavier forms HDS and D_2S; consideration is being given to the cases in which ordinary S is replaced by O or Se, or a less common isotope of S or O. These molecules, not being linear, have no axis about which they are *three-dimensionally* symmetrical, and although they are not completely rigid it has been found of value to know the rotational energy levels of the asymmetric rotor (freely rotating asymmetric rigid body). It is not necessary to work with three general values a, b, c for the reciprocal principal moments of inertia, since it has been shown by Ray,[1] as a result in matrix mechanics, that the energy is easily derived from a "reduced energy" $E(\kappa)$, where, assuming $a \geq b \geq c$, κ is a parameter of asymmetry defined by $(2b - a - c)/(a - c)$, so that $-1 \leq \kappa \leq 1$; oblate and prolate symmetry correspond to $\kappa = 1$ and $\kappa = -1$ respectively, and $\kappa = 0$ has been called the "most asymmetrical" case. The quantity $E(\kappa)$ can be formally considered as the energy of a hypothetical rotor with reciprocal moments $1, \kappa, -1$.

The values of $E(\kappa)$ have already been computed by the New (Matrix) Quantum Mechanics for $\kappa = -1(.1)1$ up to $J = 12$, where J is the quantum number associated with the total angular momentum. The values up to $J = 10$ have already been published by King, Hainer & Cross,[2] and it is stated that the values for $J = 11$ and 12 will be published shortly.

The present work aims at extending the calculation of $E(\kappa)$ to high values of J (up to about $J = 50$) by application of the Correspondence Principle. A diagram (Fig. 2) in the June report illustrates the relationship of the Correspondence Principle and New Quantum Theory levels for $J = 3$. For high J the C.P. values may be expected, in some regions of such a diagram, to lead to utilizable numerical information about the desired N.Q.T. values.

The C.P. levels are obtained from the Newtonian solution by subjecting the values of the usual cyclic integrals to quantum restrictions. It is well known that the Newtonian solution for an asymmetric rotor involves an elliptic integral of the third kind. Fortunately, only the complete integral (limits of integration 0 and $\frac{1}{2}\pi$) is required in the quantization, and this has been tabulated by Heuman[3] from a formula of Legendre. The quantum problem requires a rather complicated inversion of the table, and it was found that this and other features rendered impracticable the use of punched-card methods with IBM equipment, which it was desired to use on account of the very large number of cases to be computed. Consequently it was decided to invert Heuman's table into the form required, so that direct interpolations could then be performed by machinery. It may prevent misconception to say that no punched-card technique is discussed in the particular pair of reports under review.

A few formulae will give some idea of the inversion which has to be made. A "reduced-energy ratio" is defined by $\eta = E(\kappa)_{J,K}/J(J + 1)$, and a "quantum-number ratio" by $\lambda = K/[J(J + 1)]^{\frac{1}{2}}$. Here J and K are quantum numbers, and $K = 0(1)J$, so that $0 \leq \lambda < 1$. The problem is to find η for given κ, J, K. There are different cases according to the values of κ and η. It has been found best to derive explicit formulae for the case $\eta \leq \kappa$. The result

of the analysis is that

$$\lambda = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\cos^2\alpha \sin\beta \cos\beta \sqrt{(1 - \cos^2\alpha \sin^2\beta)}}{\cos^2\alpha \cos^2\beta + \sin^2\alpha \cos^2\phi} \frac{d\phi}{\sqrt{(1 - \sin^2\alpha \sin^2\phi)}},$$

which is exactly the function tabulated as $\Lambda_0(\alpha, \beta)$ by Heuman,[2] provided that

$$\eta = \cos 2\beta, \qquad \kappa = \frac{1 - \sin^2\alpha \tan^2\beta}{1 + \sin^2\alpha \tan^2\beta}.$$

Heuman's table has to be inverted so as to give η as a function of κ and λ.

An interval .1 was chosen for κ and λ. Since the undersigned, working with the above formulae and Heuman's tables, at first failed entirely to reproduce a few tabulated values of η selected for test, and since it is because of the numerical tables that this review appears, it is believed that a description of the tables may usefully be prefaced by the following detailed considerations. Let us suppose the arguments κ to run horizontally along the top of the table, reading $-1(.1)1$ from left to right. Let us also suppose arguments λ to run vertically along the *right* side of the table (for reasons to appear), reading $0(.1)1$ from top to bottom. For each κ, a real solution of the above equations exists (η is unique and satisfies $\eta \leq \kappa$) only when λ is not less than a certain value (depending on κ). Thus from the above equations we can calculate a triangular array of values of η lying below a zigzag dividing line (determined by the tabular intervals through which the curve $\eta = \kappa$ passes), running from the bottom left corner to the top right corner. But for given λ, it is known that η is an odd function of κ, and the triangular table may be converted into a rectangular one, in which the values are antisymmetrical about the center of the array, by providing arguments λ running vertically along the *left* side, reading $0(.1)1$ from bottom to top, for use in reading values above the dividing line, and by inserting these upper values in accordance with $\eta(-\kappa) = -\eta(\kappa)$. The left and right argument scales apply respectively above and below the dividing line, across which the values of η, but not their second and higher derivatives, are continuous.

It is clearly sufficient to tabulate half the full rectangular array described above, and neater to tabulate the left or right halves of the rectangle than to tabulate the irregular triangles below or above the dividing line. Since the important paper of King, Hainer & Cross[3] employed the range $-1 \leq \kappa \leq 0$ (because in triatomic molecules κ, for reasons there given, is more often near the prolate value -1 than near the oblate value $+1$), Table I in the June report consists of the left half; it gives η to 6D for $\kappa = -1(.1)0$; a scale of $\lambda = 0(.1)1$, reading upwards, is given on the left, and no λ-scale appears on the right. Thus anyone wishing to use the above formulae, which as they stand apply only below the dividing line, to check a value in the table should remember (a) below the dividing line, to use 1 minus the indicated λ, (b) above the dividing line, to reverse the signs of both κ and η. Table I of the August report gives the same values, with two additions. In the first place, a right-hand scale of arguments λ is given, as far as is necessary for the half-table, for use below the dividing line; this simple little addition is very welcome, and makes caution (a) unnecessary. Secondly, values of $-\partial\eta/\partial\lambda$ and $\partial^2\eta/2\partial\lambda^2$, for use in interpolation, are given to 6D below every value of η (the signs are to be prefixed according to the instructions on the previous page).

There are two subsidiary tables. Table II (June) gives, for $\kappa = -1(.1)1$, coefficients in the expansion of η in powers of $1 - \lambda$, up to the coefficient of $(1 - \lambda)^6$, to 8–9D. The coefficients of the first and second powers of $1 - \lambda$ agree numerically (to 6D) with the values of $-\partial\eta/\partial\lambda$ and $\partial^2\eta/2\partial\lambda^2$ given in the top and bottom rows of triplets in Table I (August). The values in Table II were obtained by expanding an algebraic form of the integral for λ round $\eta = -1$, and inverting series. Table III (August) gives information about maximum errors in interpolating Table I; the greatest errors occur near the dividing line ($\eta = \kappa$), and are unimportant because in this region the Correspondence Principle completely ceases to be useful.

It is pleasant to see Heuman's fine table occupying such a vital place in an important quantitative investigation. It is impossible to say whether King's somewhat special in-

version of it will find any application other than that for which it was made. Certainly, like the tables of King, Hainer & Cross, it is a purely mathematical table arising in a context which will interest many applied mathematicians, and it appears from the reports to have rendered good service in the analysis of several band spectra. A few misprints in mathematical formulae will doubtless disappear in the printed version.

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[1] B. S. RAY, "Über die Eigenwerte des asymmetrischen Kreisels," *Zeits. Physik*, v. 78, 1932, p. 74–91.
[2] G. W. KING, R. M. HAINER & PAUL C. CROSS, "The asymmetric rotor. I. Calculation and symmetry classification of energy levels," *Jn. Chem. Phys.*, v. 11, 1943, p. 27–42.
[3] C. HEUMAN, "Tables of complete elliptic integrals," *Jn. Math. Phys.*, v. 20, 1941, p. 127–206, 336.

468[M, P].—J. G. FREEMAN, "Mathematical theory of deflection of beam," *Phil. Mag.*, s. 7, v. 37, Dec. 1946 (publ. July 1947), p. 855–862. 17 × 25 cm.

The five 2D tables on p. 858 are, for $\alpha = 0(5^\circ)180^\circ$, of

$$P(\alpha) = \int_0^\alpha \sin^{\frac{1}{2}} t\,dt = 2\int_{\frac{1}{2}\pi-\frac{1}{2}\alpha}^{\frac{1}{2}\pi} (1 - 2\sin^2 u)^{\frac{1}{2}} du,$$

$$Q(\alpha) = 2\cos\alpha \sin^{\frac{1}{2}}\alpha + \sin\alpha P(\alpha),\quad S(\alpha) = 2\sin\alpha\sin^{\frac{1}{2}}\alpha - \cos\alpha P(\alpha),$$

$S(\alpha)/Q(\alpha)$, and $Q^2 \cos\alpha$. In the last two tables the values are also given for $\alpha = 171^\circ, 172^\circ, 176^\circ, 177^\circ$.

469[N].—JOH. HAGE, *Bond Value Tables*. The Hague, Nijhoff, 1946. xii, 242 p. + folding plate. 18.7 × 26.7 cm. 20 gulden, clothbound.

This book is an edition in English of a set of tables published in the Dutch language, copyrighted in 1939. As stated in the introductory general notes, the main problems to be treated by these tables are of two types: A. Computation of the present value (hereafter called the value) of a bond or loan at a given yield rate, B. Computation of the yield rate for a given value. Four types of loans are considered by the four main tables: I. Irredeemable loans, p. 1–2; II. Loans redeemed in a single payment, p. 3–67; III. Loans redeemed in a series of equal payments, p. 69–133; IV. Loans redeemed by annuities, p. 135–199. The presence of Tables III and IV is a distinguishing characteristic of this book and is sufficient to justify its usefulness.

These four tables have the following form. The present value at yield rate of the loan occurs in the body of the table on the basis of 100, being entered to 2D. The coupon rate, called the nominal rate and quoted on a yearly basis compounded semiannually, is entered across the top, ranging from 2½% (3% T. II–III) to 7% at ¼% intervals. The yield rate, compounded annually, is listed down both sides of the page, ranging from 2% to 8% at intervals of ¼%. Also on the right hand side of each of Tables II, III are two additional columns. One, headed PP, gives to 3D, the differences in value for a 1% change in the coupon rate for each yield rate listed. The second column, headed C, in T. II–III, gives for each yield rate listed an entry for computing the reduction in value caused by a 1% coupon tax. This column, as explained in the illustrative examples, is also convenient to use for computations involving loans redeemed at a premium. This latter use would seem to be the only reason for the presence of the C column in the present edition. In Tables II, III, IV each page contains the entries for a specified year of maturity of a loan, and the time interval is 1(1)60(5)80 years.

In addition to the four main tables there are three others. Table IVa, p. 200, provides the entries for calculation of the reduction in value due to a coupon tax on an annuity loan, supplanting column C of Tables II, and III. Table V, p. 201–202, gives entries for computing

corresponding rates of interest. Table VI, folding plate, provides factors for the computation of accrued interest between dividend dates.

Throughout the book the typography is excellent. The table entries are legible and well-spaced. Table VI is folded in such a way that it may be drawn out for convenient reference while the other tables are in use.

A comprehensive set of examples is given on the use of the tables for problems in which the solution is not read directly from the tables. These include for each of the four cases the following computations: of value if the yield rate is not tabulated, of value if the coupon rate is not tabulated, of yield rate with coupon tax considered, of yield rate for redemption at a premium (not for Table I), of yield rate if value is not tabulated, of value at intermediate dates, of yield rate for fractional time intervals, and of parity value for two loans. Two general types of problems are illustrated in the computation of value for deferred amortization and of value for loans with irregular amortization.

A final section of the text gives a brief description of the formulae used in the computation of the table entries. It is stated here that Tables I to IV have been calculated twice and in particular Tables II and III by two different formulae. Furthermore all four tables have been checked by a summation procedure. A similar check was also used to verify the proof sheets from the manuscript. This would seem to guarantee a very high degree of accuracy for the table entries.

An objection to the form of the tables might be in the use of yield rates on an annual basis rather than a semiannual basis as is customary. Also it would have been entirely feasible and an added convenience to have at least one of the columns of yield rates in decimal form for use in interpolation. A few minor typographical errors in the textual material may be noted; for example: on p. vii, last line, *for* 250, *read* 205; on p. 206, 1.–4, the spacing is omitted in "coupon dates"; p. 214, 1.–2, an extra *t* is inserted; p. 239, 1.–5, "an summation" should read "a summation." Some of the terms used, especially in the section describing the formulae for computation of the table entries, do not follow standard usage closely and make it a little difficult to follow the development. In particular two items are out of place in an edition for use in the United States. In the first place, since the coupon tax is not in use here, the emphasis on computations involving such a tax is not needed. In the second place, the fractional form given on page 242 for the computation of time between dates is based on European usage in which, for example, April 23, 1947, is written $19\frac{23}{\text{IV}}47$, and it is not as familiar or convenient as the usual year, month, day form of subtraction for dates. The illustrative examples are clear and informative and do much to increase the possible range of usefulness of the book.

Of the four tables appearing in the book, the material of Table II is most available in other references. Such standard references are (a) FINANCIAL PUBLISHING CO., *Monthly Bond Values*, second ed., Boston, 1941 (see *MTAC*, v. 1, p. 114–115), which contains entries to six decimal places for time intervals of one month and (b) D. C. JOHNSON, C. STONE, M. C. CROSS, & E. A. KIRCHER, *Yields of Bonds and Stocks*, New York, Prentice Hall, 1923, which gives yield rates in the body of the table to three decimal places and values to the nearest half unit on the basis of 100. For example, for a 10 year 5% semi-annual bond to yield 6% semi-annually, the *Monthly Bond Values* shows a value of 92.561263. To find the value in the present tables it is first necessary to convert to an equivalent yearly rate of 6.090% and then to interpolate, obtaining 92.56. For amounts in excess of $1000 two decimal places would not be sufficient to give values to the nearest cent. In comparison with the second reference above, Table II of the present book would give more accurate values and equally accurate yield rates.

Tables III and IV are not duplicated in frequently used references and can be very useful for particular problems. Serial bonds fall naturally into the category treated by Table III. It is of interest to note that the annuity loans treated by Table IV are of the type wherein "the portion of the annuity relating to the repayment of capital is payable yearly and the portion relating to interest is payable half-yearly."

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470[Q].—JULIUS EBSEN, *Azimut-Tabellen enthaltend die wahren Richtungen der Sonne, des Mondes und anderer Gestirne deren Declination 29° Nord oder Süd nicht überschreitet für Intervalle von 10 Zeitminuten zwischen den Breitenparallelen von 0° bis 30° Nord oder Süd.* (*30° bis 72° Nord oder Süd.*) Eighth edition, Hamburg, Eckardt & Messtorff, 1940. Two v.: iii, 124 p. and v, 172 p. (numbered 120–291). 19.5 × 27.5 cm.

These two volumes are a reprint of a table which has gone through many editions. It will be noticed that the pagination is continuous, with an overlap of four pages to cover the repetition of latitude 30°. The fifth edition was published in 1914, the fourth in 1909, the third in 1903, and the original edition dates from 1899 (xiii, 291 p.). The tables are not listed in FMR, *Index* (but neither are H.O. 71 and 120) and do not seem to be widely known outside Germany.

The tables give azimuth, to 0°.1, for the three arguments: latitude 0(1°)72°, declination ± 0(1°)29° and hour angle $0^h00^m(10^m)8^h00^m$ or some smaller limit depending on the time of rising or setting. Azimuths are not tabulated when the body concerned is within about 13° of the zenith (here interpolation ceases to be linear, for the azimuth to have much meaning) or below the horizon. The limitation to within 8^h of the meridian is very odd and can only have been made to simplify the pagination scheme. At present four pages are devoted to each latitude, with 49 lines on each page covering 8 hours of hour angle; with 15 columns on each page, the first opening is devoted to declinations of the "same" name and the second to declinations of "opposite" name. At the top of each column, for declinations of 24° or less, are given the hour angle of rising and setting and the corresponding azimuths. One curious feature is that the hour angle of rising is measured from the antimeridian, whereas, in common with modern tables of this nature, the hour angle used as argument is measured from the meridian. This is undoubtedly a relic from earlier editions in which a double argument of hour angle "a.m. or p.m." was used.

Although the tables have certainly been reset since the earlier editions, the format, type and style remain the same. The printing, paper and binding are excellent, but the type used is not the most suitable for tables. Between the fifth and the eighth editions, the rules for the sense of the azimuth (printed on each page) have been reworded to accord with the new argument. The many pages of illustrations and examples in the fifth edition have been dropped in favour of one page of notation and examples, and one page of compass courses.

The tables cover a range which is already catered for by many other well-known tables; no detailed comparison has been made, but it is unlikely that there are many errors.

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471[Q].—GERMANY, DEUTSCHE SEEWARTE, *Azimutdiagramme für alle Breiten, Deklinationen und Stundenwinkel*, prepared for the Oberkommando der Kriegsmarine. Second edition, Hamburg, 1944, vii, 331 p. (some blank) + flap. 27.5 × 28.3 cm., bound.

As the title announces, this book of diagrams is designed to give azimuths of celestial bodies for all combinations of latitude, declination and hour angle. The diagrams take the form of declination curves drawn against a rectangular grid of hour angle and azimuth; separate diagrams are given for each degree of latitude up to 80° and thereafter for 82°, 84°, 86°, and 88°. The hour angle scale is such that the six hours arranged vertically on each page occupy 8.4 inches; with 1.4 inches to the hour, one minute is rather more than 0.02 inches and the scales on each side of the page are actually subdivided to minutes. The horizontal azimuth scale covers 90° on each page in 7.9 inches (exactly 20 cm.) giving nearly a tenth of an inch to a degree. The rectangular grid consists of pecked lines for every five minutes of hour angle (just greater than 0.1 inches) and every degree of azimuth; every

half-hour and each 10° is drawn continuously, while the 5° lines are emphasized. Printed scale divisions are provided in clear type on both sides and on the top and bottom.

The declination curves, drawn for each degree, stand out prominently against this background; every fifth curve is labelled and is heavier than the others, with the 10° curves still more emphasized. The distance between successive declination curves varies considerably with azimuth and latitude; curves for individual degrees are dropped when the density becomes too high—for instance near hour angles of 0^h and 12^h.

The curves are bounded above by a pecked curve labelled "Höhe = 80°" (altitude = 80°) and below by a similar curve corresponding to rising or setting, i.e., to altitude 0°. The two sets of curves on either side of the zero declination curve are clearly marked as being of the same name or of opposite name to the latitude; all declinations from 90° "same name" to the limit 90° − latitude "opposite name" are given.

The upper bounding curve passes through the points 0^h, 0° and 0^h, 180°; the lower curve passes through the points 12^h, 0°; 6^h, 90°; 0^h, 180°. It is thus seen that the diagrams for each latitude can always be put on three pages, though in high latitudes two of these can be combined.

The preface suggests an accuracy of reading to 0°.1 or 0°.2, and this can doubtless be achieved with care in some areas. Interpolation for latitude is performed numerically, the variations for a degree of latitude being printed in red at frequent intervals on the declination curves; an auxiliary table (on the flap) caters for proportional parts up to a difference of 5°.7, the correction so formed being applied to the azimuth read from the diagram. Instructions are given on each page for converting diagram azimuths to true azimuths.

The diagrams essentially give the complete relationship between four consecutive parts of a spherical triangle, and thus can be used, within their obvious limitations, for the solutions of all problems involving the expression of one of the four parts in terms of the other three. Advantage is taken of this to apply the diagrams to the solution of a whole range of navigational problems, and elaborate rules for these are tabulated and exemplified. In particular, it is suggested that the diagrams can be used for determining altitude, after the azimuth has been found; to assist in this, and other problems, detachable auxiliary scales are provided for converting the azimuth scale to time, and the hour angle scale to arc.

The whole volume has been prepared with much care; the drawing and reproduction of the diagrams is in keeping with the finest German technical standards, while the paper and binding are excellent. Although a second edition, it is thought that the earlier, 1941, edition was confined to latitudes 54°–65°.

According to the preface, the values of the azimuth were taken from published tables for latitude and declinations less than 70° and computed thereafter.

What are the advantages of graphical presentation over tables, and how do these diagrams compare with H.O. 71 and 120, BURDWOOD, DAVIES, CUGLE, EBSEN and other well known azimuth tables? In the reviewers' opinion the relative advantages are nicely balanced: the diagrams are less bulky and interpolation is easier for hour angle and declination; the chances of error in using the tables are less especially in conditions of poor light. The diagrams are excellent of their kind, whereas none of the tables is first class; poorly produced diagrams would rapidly lose their advantage, even over poorly produced tables. In astronomical navigation both numerical and graphical methods must be used, and it is largely a matter of personal preference which method is used for the azimuth.

J. B. PARKER & D. H. SADLER

472[U].—GERMANY, REICHSLUFTFAHRTMINISTERIUM, *Höhentafeln nach Sternzeit für die Breiten 50° bis 56° N 1944–1945*. Berlin, 1944, 200 p. 19 × 26 cm. Not available.

These tables were designed for the reduction of star sights in the German Air Force. The altitudes and azimuths of twelve selected stars are given for four integral latitudes, N. 50°, N. 52°, N. 54°, and N. 56°, the argument being hours and minutes of sidereal time. The excellent principle of arrangement is similar to that first proposed by Commander C.

H. HUTCHINGS, U.S.N., in 1942 and subsequently used experimentally in the *Experimental Astronomical Navigation Tables*[1] produced for the Royal Air Force in 1943. The same principle is used in *Star Tables for Air Navigation* (H.O. 249; see RMT 473).

As in all German Air Force publications, sidereal time is here defined with reference to the transit of the first point of Aries across the anti-meridian; it thus differs by 12^h from sidereal time as normally defined. Discussion of the merits of this definition, which brings the measurement of sidereal time into line with mean time, is outside the scope of this review.

The tables are arranged in sections according to latitude, and there would appear no reason why more, or less, than four latitudes should not be bound together. Each opening contains the tabulations for an hour of sidereal time, at an interval of one minute; excellent thumb indexes for latitude and hours enable the appropriate opening to be found readily. At each opening are given the altitude, to the nearest minute, and the azimuth, to the nearest degree, of twelve stars, together with the "Pole Star Correction" and the azimuth of *Polaris* (to the nearest half degree). Six stars are assigned to each page, the *Polaris* data being given on the inside page margins; changes of stars are made only at integral hours.

Although the layout of the tables is good, various points indicate lack of experience of the presentation of tabular matter. The general appearance is marred by irritating details, such as the senseless repetition of degree and minute signs for each entry, the use of a vertical rule to indicate that the azimuth takes the same value for several consecutive entries, and too many horizontal and vertical rules; the 61 lines on each page are divided by heavy rules on each side of the multiples of 10° and by light rules on each side of the odd multiples of 5°. Modern style figures (without heads or tails) are used, to the undoubted detriment of easy legibility—of the first order of importance in the air. This feature is emphasized by the use of bright yellow paper (used also in the German Air Almanacs I and II) which does in fact slightly help legibility under conditions of poor illumination. The printing is good and the binding adequate. No errors have been found in a casual examination.

A statement on each page indicates that the tables are valid for the two years 1944 and 1945. If the star positions used were for the beginning of 1945, as is indicated by independent computation, the maximum error due to the combined effects of precession, nutation, and aberration will not exceed 1′ in the period concerned; the tables could be used for a further year without serious loss of accuracy. Since no annual corrections are given, it is clearly intended to reprint the table every two years. It is not known whether earlier or later editions exist.

The actual arrangement of the stars is curious. Each of the two groups of six is arranged alphabetically in order of star names, but the reason for allocating one star to the right-hand page, and another to the left, is not clear. On the whole, the "best" stars (from the point of view of altitude) appear to be given on the left-hand page, but this is not always the case. In some instances there is difficulty in selecting twelve suitable navigational stars, and some very low altitudes are tabulated: *Procyon* is listed for N. 50°, 1^h00–2^h00 S.T., even though its altitude range is 9° 58′ to 0° 28′. No star magnitudes are given.

The minimum explanation and instructions are given with the tables—and no illustrations of their use. Three small auxiliary tables are printed on the first page: the first gives mean refraction for various heights, the second the Coriolis correction to bubble sextant readings for the mean latitude, and the third is a conversion table from minutes of arc to kilometers. The refraction table confirms that refraction has not been included in the computed altitude. The explanation suggests that the tables were primarily intended for use with a timepiece keeping Greenwich sidereal time ($+12^h$); failing this, provision seems to have been made, in the shape of canvas pockets, for the insertion of conversion tables from mean to sidereal time.

No interpolation table for altitude (maximum difference 15′) is provided, as an assumed position, making the local sidereal time an integral minute, can always be used.

It is tempting to go beyond the scope of this review and to assess the value of these tables for air navigation. It will suffice here, however, to say that their chief disadvantage is that the method cannot be used for Sun, Moon and planets, and that therefore some other

method must co-exist; whether the saving on the stars outweighs the disadvantages of learning two methods and having two sets of tables, is not within our province.

J. B. PARKER & D. H. SADLER

[1] The full title of this eight-page pamphlet published by H. M. Nautical Almanac Office in 1943, "for official use only", is *Experimental Astronomical Navigation Tables for Latitude North 53° (for use between N. 52°30′ and N. 53°30′) and Time Scale Setting Data for October–November 1943*, 16.8 × 24.5 cm. The fundamental principle underlying these tables was conceived by Wing Commander E. W. ANDERSON and developed by Squadron Leader A. POTTER.

EDITORIAL COMMENT: Professor C. H. SMILEY has contributed the following review: This leaflet was prepared so that the method represented by it could be compared with other methods in common use. For each five minutes during the night in England and Western Europe, altitudes to the nearest minute of arc and azimuths to the nearest degree are given for six of the following twelve stars: Aldebaran, Alpheratz, Altair, Arcturus, Capella, Deneb, Dubhe, Pollux, Procyon, Regulus, Sirius and Vega. Altitudes as low as 6°14′ and as high as 75°17′ are given in order to have the available stars well distributed in azimuth.

The tabulated altitudes include a correction for atmospheric refraction at 20,000 ft. elevation. So long as the plane remains above 5,000 ft. elevation and the celestial body has an altitude greater than 12°, the additional correction required to take complete account of refraction is less than a minute of arc. Opposite certain sequences of altitudes of particular stars appears a heavy black line, warning the navigator that the star is near the meridian and the altitude can no longer be considered to change in a linear fashion with time; in such a case the observed altitude can be used as the argument instead of the time of observation.

Beside the altitudes and azimuths of six stars, for each five minutes of the night, the azimuth of Polaris is given as well as the Q-correction which, added to the observed altitude of Polaris, gives the latitude of the observer.

The argument in the tables is $(T - t)$, where T is an auxiliary scale-time which, over-printed on the plotting charts, replaces the longitude scale. This device allows the use of a mean-time chronometer without the usual transformation from mean time to sidereal time.

Beside the principal table of altitudes and azimuths, there is one giving scale-time settings and time-difference corrections allowing the table to be used on any night during the two months of October and November 1943.

The method appears to be one of local value, depending on the availability of the necessary over-printed charts. It would be an expensive system to maintain continuously for all latitudes and longitudes. It has the advantage that the navigator can look ahead and see what stars are going to be available at a particular time and place.

473[U].—U. S. HYDROGRAPHIC OFFICE, *Star Tables for Air Navigation. Computed Altitude and true Azimuth for all Latitudes. Preliminary edition.* (H.O. no. 249.) Washington, D. C., U. S. Government Printing Office, 1947. viii, 322 p. 23.7 × 30.2 cm. For sale by the Hydrographic Office and by the Superintendent of Documents, Washington, D. C. $2.00; foreign price, postage extra.

The preface of this volume specifically credits Commander C. H. HUTCHINGS, U. S. Navy, with having "conceived and designed" these tables, "as a rapid method of determining computed altitudes and azimuths of prominent fixed navigational stars." Tables of this character were independently suggested at about the same time by two Americans, GEORGE G. HOEHNE and C. H. Hutchings (see RMT **313**, and below).

These tables give the altitude to the nearest minute of arc and the true azimuth to the nearest degree for six stars for each integral degree of the Local Hour Angle of the March Equinox (LHA ARIES or LHA ♈) for integral latitudes 69°S to 69°N. For integral latitudes in the polar regions, 70°N–89°N and 70°S–89°S, similar information is given except the interval of the argument, LHA ♈ is 2° rather than 1°. The six stars are chosen from a list of 38, half of which are brighter than the second magnitude and the remainder are of the second magnitude. At intervals of 15° of LHA ♈ (30° in the polar regions), the table is interrupted and new stars are introduced to replace those which have moved too high or too low for good observing. The stars are listed in order of increasing azimuth, left to right, even though this means that a star must be moved to the right or left as a 15° break is crossed.

The altitudes ordinarily range from about 20° to 70°, although in some cases, altitudes down to about 10° and up to 85° are included. For example, for latitudes 15°N and 41°N, Aldebaran and Capella respectively are given down to altitudes 10°34′ and 10°41′, and for latitude 39°N, an altitude of 85°11′ is given for Vega. The altitudes have been corrected for atmospheric refraction at 5000 feet above sea level and an auxiliary table, I, provides corrections for other elevations at 5000 ft. intervals up to 40,000 ft.

Mean star places for 1948 have been used, and *average* corrections for precession have been provided in Table II for the years 1947–1951. It is precession which will make this type of table expensive; after a brief interval the table will have to be recomputed and reprinted, or elaborate tables provided to take care of the changes due to precession.

Other auxiliary tables included are: Table III, for the conversion of arc to time; Table IV, lists of the 38 stars, alphabetically arranged and in order of declination; Table V, corrections to be added to sextant altitudes of Polaris to obtain latitudes; Tables VIa and VIb, corrections for the motion of the observer for ground speeds up to 450 knots; Tables VIIa and VIIb, Coriolis corrections for a similar range of speeds.

In the principal table, all of the data for a single latitude in the interval, 69°S–69°N, are contained on two facing pages. Each page has two vertical sections of six columns each, covering an interval of 90° in LHA ♈. Each column is broken into six segments of 15°, with a star name at the head of each segment. For the polar latitudes, all of the data for a given latitude are on a single page and the blocks cover 30° of LHA ♈. The convenience and practicality of this arrangement, showing which stars will be available for observation at a given time and place, and their altitudes and azimuths, can hardly be exaggerated.

The Air Almanac may be dispensed with, providing a sidereal chronometer reading in degrees and minutes of arc is available. The advantages of such a chronometer were pointed out by AQUINO in 1933. To date, only a limited number have been made and they are quite expensive.

In the choice of stars, great emphasis appears to have been placed on having the stars well distributed in azimuth and considerably less emphasis placed on continuity. If an observation of a star happens to fall just outside of the time interval for which altitudes and azimuths are given, the navigator will be obliged to discard the observation or to resort to the uncertain process of extrapolation. For this reason, "ends" might reasonably have been kept to a minimum.

The data for latitude 42°N have been examined in this connection; it is found that 25 of the 38 stars appear on the two pages. Nunki and Marfak each appear in only a single 15° block, introducing altogether 8 ends. Rasalague and Rigel each appear in two separate single blocks. By removing one block for Vega (LHA ♈ 285°–299°) in which the altitudes range from 74°18′ to 84°15′, and replacing it by one for Rasalague, one would have three consecutive blocks instead of two isolated ones, and four fewer ends. Likewise if the blocks for Rigel (LHA ♈ 30°–44° and 90°–104°) were replaced by similar blocks for Aldebaran and Betelgeuse respectively, six ends would be eliminated. It is appreciated, however, that no two people would make precisely the same choice of stars. Undoubtedly a great deal of time and thought went into the choosing of stars and unquestionably good arguments could be offered for the choices made.

The tabular values in this volume should establish a high standard of accuracy since they were computed by experts at the NBSCL. Altitudes of Vega near the meridian for latitudes 65°N to 74°N appear to be out by 1′, probably due to an erroneous correction for refraction.

Although the paper and the plastic binding are good, the printing is not up to the excellent quality usually provided by the Hydrographic Office. The material is too crowded and the type too small. Even if the volume has to be expanded into two, one for the northern hemisphere and another for the southern, an effort should be made to improve the legibility of the material. The printing appears to have been done by photo-offset or some similar process from sheets turned out by the automatic machines of the NBSCL. Even so, an occasional digit fails to appear in print, for example, the terminal digit of the azimuth of Fomalhaut near the meridian in 25° south latitude.

It is true that an aeroplane can many times rise above the clouds and observe stars at altitudes 20° and greater. However, no provision has been made for the case where the only stars available are at altitudes less than 10°. Although altitudes of 10° or less may be uncertain due to refraction, the corresponding azimuths are reliable. With the rapidly increasing speeds of planes, one may well look ahead to the time when it will again be feasible to "steer by a star"; stars of low altitude will be very useful for this purpose.

To return briefly to the development of these tables, Hutchings' specific recommendations were published in 1942[1] and comments by AGETON,[2] WEEMS,[3] and AQUINO[4] appeared in the next two years. In addition, other individuals upon request sent their comments on the proposed tables directly to the Hydrographic Office. It is interesting to note that in Hutchings' original proposal, the interruption of a given column to introduce a new star might be made at any point; altitudes were to be given to the nearest 0′.1 and azimuths to the nearest 0°.1. In a description by Commander W. J. CATLETT in 1945 of the forthcoming tables,[5] the 15° break had been introduced, altitudes were given to the nearest minute of arc and azimuths to the nearest degree.

Although several other tables of this general design have appeared earlier, this volume is apparently the only one in which *all* latitudes have been covered in a single volume. One may reasonably hope that another volume (or volumes) of tables of modern design allowing one to use observations of the sun, moon and planets may also be provided in the near future, perhaps as H.O. no. 250.

CHARLES H. SMILEY

[1] C. H. HUTCHINGS, U. S. Naval Inst., *Proc.*, v. 68, 1942, p. 1279–1284.
[2] A. A. AGETON, U. S. Naval Inst., *Proc.*, v. 68, 1942, p. 1303.
[3] P. V. H. WEEMS, U. S. Naval Inst., *Proc.*, v. 68, 1942, p. 1760–1761.
[4] F. R. DE AQUINO, U. S. Naval Inst., *Proc.*, v. 70, 1944, p. 315–318.
[5] Institute of Navigation, *Minutes of New England Regional Meeting* . . . 27 Aug. 1945, offset print, p. 7.

474[V].—HOWARD W. EMMONS, *Gas Dynamics Tables for Air*. New York, Dover Publications, 1947. 46 p. 15.4 × 26.5 cm. $1.75. "The author wishes to acknowledge his indebtedness to Mr. J. ARTHUR GREENWOOD, who very ably carried out all the numerical computations required to produce these tables" (p. 16; Tables I–IV, p. 17–36).

"*Introduction.* Recent trends in the development of high-speed aircraft render increasingly erroneous calculations based upon the assumption of an incompressible fluid. There is no need, however, to look to the future for justification of tables such as these, for ballistics already deals with velocities several times that of sound."

"The speed of sound that enters as the significant reference velocity in all discussions of compressibility effects on the flow of fluids is for air a function of temperature given in Table IV. At the standard sea level temperature of 59F the speed of sound is 1116.4 ft/sec, 761.2 miles/hour or 340.3 meters/second."

T. I: *Isentropic Gas Dynamics Functions for Air*, p. 17–30. For the MACH number, $M = V/a$, V (velocity), a (speed of sound), $M = 0(.001).05(.01).8(.001)1.2(.01)2(.1)5(1)25$, the values are given, 3 to 5S mostly, for the following 8 functions:

$T/T_0 = [1 + \frac{1}{2}(\gamma - 1)M^2]^{-1}$, $\gamma = 1.4$ (isentropic exponent), T (temperature);
$p/p_0 = (T/T_0)^{\gamma/(\gamma-1)}$, p (pressure); $\rho/\rho_0 = (T/T_0)^{1/(\gamma-1)}$, ρ (mass density);
$a/a_0 = (T/T_0)^{\frac{1}{2}}$; $V/a_0 = aM/a_0$; $\rho V/\rho_0 a_0 = (\rho/\rho_0)(V/a_0)$;
$\rho V^2/2p_0 = \frac{1}{2}\gamma p M^2/p_0$; $A/A^* = (\rho^*/\rho_0)(a^*/a_0)/(\rho/\rho_0)(V/a_0)$,
$\rho^*/\rho_0 = [2/(\gamma + 1)]^{1/(\gamma-1)} = .63394$, $a^*/a_0 = .91287$; * indicates the critical condition; i.e., the condition in which the fluid velocity equals the local speed of sound. Graphs p. 37–40.

T. II: *Gas Dynamics Functions for Normal Shocks*, p. 31–33. For $M_1 = [1(.01)2(.1)5(1)25; 5S]$ are given the values of the following 8 functions:

$p_2/p_1 = 2\gamma M_1^2/(\gamma + 1) - (\gamma - 1)/(\gamma + 1)$; $V_2/V_1 = 2(\gamma + 1)^{-1}(V_1/a_*)^{-2}$;
$\rho_2/\rho_1 = V_1/V_2$; $T_2/T_1 = (p_2/p_1)/(\rho_2/\rho_1)$;
$M_2 = [2/(\gamma + 1)]^{\frac{1}{2}}/[(p_2/p_1)(T_2/T_0)]^{\frac{1}{2}}$; $P_{20}/p_1 = (P_{20}/p_2)(p_2/p_1)$;
$p_{2_0}/p_{1_0} = (p_{2_0}/p_1)(p_1/p_{1_0})$, 1_0 indicating isentropic stagnation condition immediately before a shock wave, 2_0 indicates isentropic stagnation condition immediately behind a shock wave. Graphs p. 41–44.

T. III: *Characteristics Table*, p. 34–35.
$M = \csc\alpha$ or $\alpha = \sin^{-1}(1/M)$,
$\omega = [(\gamma + 1)/(\gamma - 1)]^{\frac{1}{2}} \tan^{-1}\{[(\gamma - 1)/(\gamma + 1)]^{\frac{1}{2}} \cot\alpha\}$;
for $\nu = \omega + \alpha - \frac{1}{2}\pi = 0(.5)130$ are given the values of α, 2D, and of M, 3–5S; graphs p. 45.

T. IV: *Acoustic Velocity-Temperature Table*, p. 36.
For $T = 300(2)698$ are given the values of $a = 49.019T^{\frac{1}{2}}$ ft./sec., 4–5S; graph, p. 46.

Extracts from text

475[V].—MASSACHUSETTS INSTITUTE OF TECHNOLOGY, Department of Electrical Engineering, Center of Analysis, Technical Report no. 1, work performed under the direction of ZDENĚK KOPAL, under NOrd Contract No. 9169: *Tables of Supersonic Flow around Cones*. Cambridge, Mass., 1947, xviii, 558 p. + 9 folding plates. 20.4 × 26.8 cm. For sale by Library of Congress, Washington, D. C., $3.50.

The general equations of fluid mechanics have been known for over a century. These equations, including the continuity equation, the NAVIER-STOKES momentum equation, and some form of energy equation, are adequate to solve problems of flow of fluids with viscosity through channels of arbitrary shape or around bodies of arbitrary shape. Many special solutions are known. Most of them, however, are for the motion of a perfect fluid (that is, one without viscosity or heat conduction), around some simple geometric shape. The majority of exact solutions available to date concern the flow of an incompressible fluid, that is, a fluid whose density is constant. For the motion of air, such incompressible fluid solutions are adequate, provided the fluid velocities are so low that no appreciable changes of density occur. The number of engineering problems in which the fluid velocity became high enough to include the effects of compressibility were limited to a few applications in steam turbine design and projectile design, until the speed of aircraft became so great that the compressibility problems became significant in that field. During the 1930's, the speeds of aircraft became so high that serious compressibility effects became apparent and thus necessitated the search for additional solutions to the problems of the flow of air about bodies of various shapes. The development of rockets and the probable development in the near future of supersonic aircraft have extended the need for solutions to the equations of fluid mechanics and to the range of MACH numbers greater than unity.

The problem of the supersonic flow about a cone is a most significant one because of the fact that a sharp, pointed fusilage is an appropriate one for supersonic aircraft and rockets, just as it is appropriate for projectiles. Fortunately, the exact solution for the flow about cones is reducible to a fairly simple numerical process and the present tables are designed to supply such solutions. The fact that the infinite cone is the appropriate "simple" geometric form to consider in supersonic flow (rather than the sphere as in subsonic flow), was pointed out in a qualitative way by BUSEMANN[1] and BOURQUARD.[2] It remained for TAYLOR & MACCOLL[3] to work out the detailed equations and to obtain numerical solutions for a number of important values (mentioned later), and to check these solutions against experimentally observed flows. At the present time, the calculated flow around cones has two principal uses, (1) the predicted wave angles check so well with experiment that the

cone is now frequently used as a means of calibrating an air stream of unknown velocity; (2) many projectiles, rockets, and supersonic planes have a conical nose where these solutions can be used directly to predict the air forces encountered. Even in the case that an ogival nose is used, the computation of the air forces must start with the conical solution at the vertex. In mathematical terms, this means that the type of singularity in the solution to the supersonic flow of air over a pointed body of revolution is the same as that for the flow about a cone.

The present work presents an extensive set of tables giving a complete solution in 5D tables for the flow of air around cones at zero angle of attack. These solutions are complete in the sense that not only do they give the cone angle, wave angle, and Mach number relations, but also the velocity components and speed of sound in the region between the shock wave and the cone surface. Such results are adequate for the solution of other problems of flow about bodies of revolution when it is pertinent to start their solution with the corresponding cone solution. In a twelve-page introduction to the Tables, there is a review of the nature of the problem and the general aspects of its solution, as well as a derivation of the necessary equations. Essentially the Tables are a solution of the differential equation:

$$\text{(1)} \qquad \frac{d^2u}{d\theta^2} + u = \frac{a^2(u + v\cot\theta)}{v^2 - a^2}$$

where u is the radial velocity component
v is the tangential velocity component
θ is the colatitude
a is the velocity of sound in the gas.

The velocity of sound is given by:

$$a^2 = \tfrac{1}{2}(\gamma - 1)(c^2 - u^2 - v^2)$$

where c is the (constant) velocity of expansion into a vacuum
γ is the (constant) ratio of specific heats for the gas.

Equation (1) is subject to boundary conditions $u = u_s$ and $v = 0$ for cone semi-vertex angles $\theta = \theta_s$. For each u_s, a solution is constructed numerically toward increasing θ until the shock wave equation $\tan\theta = -\dfrac{\gamma - 1}{\gamma + 1}\,\dfrac{c^2 - u^2}{uv}$ is satisfied.

This value of $\theta = \theta_w$ gives the position of the conical shock wave attached to the nose of the cone. All of the remaining physical properties of the solution now follow immediately from well-known gas dynamics relations.

As is well known, there are in general two solutions for each initial value of the cone angle θ_s. In the present work, both solutions are given. The introduction includes an adequate description of the reason for obtaining these two solutions and notes that only the weak shock or "first" solution is observed in experiments. It is also noted that solutions exist only for cone angles: $\theta_s \leq 57°.5253\ldots$

The only previously published calculations of the supersonic flow of air about cones and, in fact, the original work presenting the theory and its experimental verification was that of Taylor and Maccoll.[3] This work, although it contains very limited tables, is the classic work on this problem and has given its name, The Taylor & Maccoll Flow, to the phenomena of flow about cones with attached shock. The tables contained in this early publication are for θ_s = [10°(10°)30°; 3D], over a range of some ten values of u_s. No complete solutions are given, however, and if these were desired they would have to be reconstructed. Hence the present work is noteworthy not only because of its more extensive character and greater accuracy but also because it gives the complete solution.

A few typographical errors are worth noting because of the incorrect meaning implied as written. (1) On page xiv of the introduction, the fourth and fifth lines from the top now

read, "In doing so we find, however, that U/a_1 is *not* a single-valued function of u_s/c." This sentence should have U/a_1 and u_s/c interchanged, because in fact two values of u_s/c correspond to each value of U/a_1. (2) On page xv, on the fourth and ninth lines from the top, there appear the words "downstream." These should read "upstream."

The Tables presented are as follows:

I. Tables of Supersonic Flow of Air ($\gamma = 1.405$), p. 1–468. Values of the square of the speed of sound and the radial and tangential velocity components are given in terms of the (constant) velocity that would be attained by the air in front of the shock wave if it expanded into a vacuum. These solutions are given for values of $\theta_s = 5°(2°.5)25°(5°)50°$, for 32 values of u_s ranging from .175 to .99551 (with increments ranging from .00051 to .05). For each complete solution, that is, for each pair of θ_s, u_s values, the corresponding wave angle and Mach number ahead of the shock are given. The large number of pages used in this table could have been reduced by two-thirds through using a finer print and closer spacing. 5D values are given which, for most of the table, are rounded from computations to 6D. Those parts which were carried out to 5D only are indicated by an *. The strong shock solutions are indicated by (S) following the Mach number.

II. In this table, p. 469–475, the physical properties of major interest from the preceding solutions are tabulated. The tabulation includes 5S values of θ_s, M, T_s/T_w, p_s/p_w, ρ_s/ρ_w, T_w/T_1, p_w/p_1, ρ_w/ρ_1, K_D, as functions of θ_s, u_s for the cases presented in Table I.

III. To facilitate the use of Table I, values of u_s are given, p. 477–479, as function of the semi-apex angle of the cone $\theta_s = 5°(2°.5)25°(5°)50°$ and the Mach number $M = 1.05(.05)4.00$.

IV. Table, p. 481–483, giving, for the same range, the semi-apex angle of the shock wave, θ_w, in terms of M and θ_s. This table is the one that has been the object of former computations and is the one that would be most frequently used by those using cones for supersonic flow measurements.

V. In a small group of solutions, the velocity immediately behind the shock wave is supersonic but changes with decreasing θ and becomes subsonic at the cone surface. This table, p. 485–487, gives values relating to these solutions as well as to the division between strong and weak shock solutions. Specifically, the individual columns indicate:

(1a) Minimum Mach number for which solutions are possible.
(1b) The value of u_s corresponding to the division between 'first' and 'second' waves.
(2a) Maximum Mach number for which the stream between the cone and the shock wave is completely subsonic.
(2b) The corresponding value of u_s.
(3a) Minimum Mach number for which the stream between the cone and the shock wave remain supersonic.
(3b) The corresponding value of u_s.

VI. Besides the physically important properties possessed by the solutions to equation (1), there are the mathematical second-order discontinuities, where $v^2 = a^2$. One such discontinuity appears beyond the strong shock; the other appears inside of the cone surface. The location of these two discontinuities are listed, p. 489–491.

VII. This table, p. 493–548, is identical with Table I except that $\gamma = 1\frac{1}{3}$. This additional table was computed to show the effect of changing γ, and this particular value was chosen because steam and other commercially important gases have values of γ in this neighborhood. This table is much less extensive than Table I. $\theta = 10°(5°)40°$, while u_s has an increment of .1, except at the highest values.

VIII. This table, p. 549–551, summarizes the results given in Table VII. The quantities listed are the same as in Table II.

IX. In this Table, p. 553–555, the change of the adiabatic constant, K, and entropy, s, across the shock wave as a function of shock strength, p_w/p_1, is given.

X. Here are nine folding plates with graphs, giving the physically important properties of

the solution in a form which will be readily usable in practical work. The graphs are good to 3D at the most, and are valuable in those places where high accuracy is not required.

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[1] VON A. BUSEMANN, "Drücke auf kegelförmige Spitzen bei Bewegung mit Überschallgeschwindigkeit." *Z. f. angew. Math.*, v. 9, 1929, p. 496–498.
[2] F. BOURQUART, "Aerodynamique—Ondes balistiques planes obliques et ondes coniques application à l'étude de la résistance de l'air." Acad. d. Sci., Paris, *C.R.*, v. 194, 1932, p. 846–848. Also *Mém. d'Artill. Franç.*, v. 11, 1932, p. 135f.
[3] G. I. TAYLOR & J. W. MACCOLL, "The air pressure on a cone moving at high speeds," R. Soc. London, *Proc.*, v. 139A, 1933, p. 278–311.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in the article "A New Approximation to π (conclusion)"; RMT **451** (Schulze), **452** (Müller, Rajna & Gabba), **453** (Prokeš), **463** (Ziaud-Din, Kerawala & Hanafi), **464** (NBSCL), **465** (Col. Univ. Press), **466** (Reiz), **469** (Hage), **473** (U. S., H. O.), **475** (M. I. T.); UMT **63** (Hayashi, Brandenburg).

118. C. GUDERMANN, "Theorie der potenzial- oder cyklisch-hyperbolischen Functionen," *Jn. f. d. reine u. angew. Math.*, v. 8–9, 1832.

In the course of reading proofs of the new Chambers' 6-figure tables the logarithmic values of sinh, cosh and tanh for the range $k = 2(.001)3(.01)6$ were compared with the first six decimals of Gudermann. The following errors were noted:

	Page	*k*	Function	*For*	*Read*
v. 8,	195	2.018	sinh	2345	6345
		2.036	tanh	1940	1949
	196	2.063	sinh diff	4445	4485
		2.081	sinh	9189	9188
	198	2.169	tanh diff	266	226
	199	2.248	cosh diff	33	23
	200	2.258	cosh diff	4279	4249
		2.258	tanh	50 6	5036
		2.284	tanh diff	980	180
	201	2.301	sinh	6.693	0.693
	202	2.353	cosh diff	4205	4265
		2.377	sinh	529	5293
	203	2.414	sinh	69	59
		2.415	tanh	4628	0628
		2.445	tanh diff	310	130
	204	2.489	cosh	9190	9100
		2.498	Argument	489	498
	205	2.506	tanh	2272	2172
	209	2.701	sinh	6.870	0.870
	210	2.759	sinh diff	4477	4377
	212	2.854	cosh	999	939
		2.882	cosh	9679	9676
		2.893	sinh diff	4399	4369
		2.898	Argument	889	898
		2.898	sinh diff	4379	4369
	304	3.061	cosh	.029	1.029
	306	3.159	sinh diff	4458	4358
	307	3.202	sinh diff	4375	4357
	308	3.251	sinh diff	4356	4355
		3.298	Argument	289	298
	313	3.506	cosh	1.122	1.221
	314	3.598	Argument	589	598
	316	3.659	sinh diff	4448	4348

	Page	k	Function	For	Read
v. 9,	96	4.460	cosh	653	635
	193	4.500	tanh	8828	8928
	197	4.721	tanh	.9990	.9999
	199	4.828	sinh		95
	203	5.01	sinh	7600	7660
		5.22–5.26	sinh diff	43432	43431
	204	5.75	tanh	612	912

L. J. C.

EDITORIAL NOTE: These Gudermann tables of log sinh k, log cosh k, log tanh k, for k = [2(.001)5; 9D], [5(.01)12; 10D], occupy p. 261–336 of the reprint, *Theorie der Potential- oder cyklisch-hyperbolischen Functionen*. Berlin, 1833. Except for k = 4.828, all of the above mentioned errors are preserved in this reprint. It may be noted that in our copy of "Crelle," v. 8, and the reprint, for k = 2.258, log tanh k, the 3 in 5036 is faint but recognizable; so also for k = 2.377.

In checking the L. J. C. errata list with the copy of Crelle 8 in the Columbia University Library we made the interesting discovery that in this copy, five of the above-mentioned errata do not appear, namely those listed above on p. 304, 307, 308 (l.2), 314, 316; also on p. 313, 1.122 has been changed to 1.222; however, inspection of the table shows that the change should have been to 1.221. It is well known that early volumes of Crelle have been reprinted, but we have never before noticed a record of corrections having been made before reprinting. Can any reader inform us as to when the reprint was made? In the reprint of pages 293–320, at least, the type was evidently reset since it differs from the original of 1832.

119. HARVARD UNIVERSITY, Computation Laboratory *Annals*, v. 4: *Tables of Bessel Functions* . . ., 1947, see *MTAC*, v. 2, p. 261.

$J_3(72.10)$, *for* −0.07253 46394 8768 415, *read* −0.07253 46394 87568 415. It seems that even an electromagnetic typewriter can err, and that the necessity for proofreading is not entirely eliminated.

J. C. P. MILLER

120. E. JAHNKE & F. EMDE, *Tables of Functions*, 1909 ed., p. 65–66; 1933, p. 122; 1938, 1941, 1943, 1945 eds., p. 49. [See also *MTAC*, v. 1, p. 391–399; v. 2, p. 26, 47, 350.]

In the table of log q there are the following five errors in each of these six editions: α = 2°55′, *for* $\bar{5}$.2096, *read* $\bar{4}$.2096; α = 3°55′, *for* $\bar{5}$.4658, *read* $\bar{4}$.4658; α = 4°55′, *for* $\bar{5}$.6635, *read* $\bar{4}$.6635; α = 5°55′, *for* $\bar{5}$.8246, *read* $\bar{4}$.8246; α = 37°50′, *for* $\bar{5}$.4693, *read* $\bar{2}$.4693.

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121. N. S. KOSHLIAKOV, [On the calculation of integrals to infinite limits by means of formulae of mechanical quadratures], Akad. N., Leningrad, *Izvestiia*, s. 7, *Otdelenie matem. i estestvennykh Nauk*, v. 7, 1933, p. 802. See *MTAC*, v. 1, p. 361.

The zeros (x_i) and Christoffel numbers (A_i) for the Laguerre polynomial of the fifth degree are here given to 7D. The following corrections should be noted:

	For	Read		For	Read
x_1	0.2635581	0.2635603	A_1	.5217595	.5217556
x_2	1.4134042	1.4134031	A_2	.3986673	.3986668
x_3	3.5964256	3.5964258	A_3	.0759361	.0759424
x_4	7.0858108	7.0858100			
x_5	12.6408013	12.6408008	A_5	.0000233	.0000234

A_4 is given correctly as .0036118.

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Editorial Note: All of these corrections agree with the 8D values given in SRE/ACS 82, RMT **252**, *MTAC*, v. 2, p. 31. The corresponding 7D values which Koshliâkov gives for $\log x_i$ and $\log A_i$ must also be amended; for example: *for* $\log x_1 = 9.4208764$, *read* 9.4208800; *for* $\log A_1 = 9.7174704$, *read* 9.7174672. Similarly for the 5D table of $A_i f(x_i)$, where $f(x) = x/(1 - e^{-2x})$.

122. William Spence, *An Essay on the Theory of the Various Orders of Logarithmic Transcendents*, London and Edinburgh, 1809. Also in Spence, *Mathematical Essays*, 1819 and 1820. See *MTAC*, v. 1, p. 457–459; v. 2, p. 180.

On p. 63 of the 1809 edition and p. 64 of the 1819 and 1820 editions is a table of the values of the function $C_1(x)$ which is apparently the first table of $\tan^{-1} x$; $x = [1(1)100; 9D]$. Comparison with the NBSMTP *Table of Arc Tan x* reveals the following errors in Spence: $x = 38$, *for* 1.54448 7135, *read* 1.54448 6609; $x = 43$, *for* 1.54754 4702, *read* 1.54754 4703; $x = 61$, *for* 1.54440 4352, *read* 1.55440 4352. Since Spence did not round off his tables, the corrected values are not rounded off.

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123. J. W. Wrench, Jr., *Values of Steltjes' sums* $S_k = \sum_1^\infty n^{-k}$.

In the table of the final 7-digit terminal figures in 37D values of S_k, $k = 2(1)33$, *MTAC*, v. 2, p. 138, Dr. Wrench has erred in the value 6043727 given in connection with S_{21}. I find beyond the 30th digit 6043730 459. With this correction, the checking relation (using 37D values of S_k) now yields a discrepancy of only 0.5×10^{-37} instead of the previously noted (p. 138) 3.5×10^{-37}.

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UNPUBLISHED MATHEMATICAL TABLES

63[A, B, D, E].—J. W. Wrench, Jr., *A New Table of* $\pi^n/n!$ Manuscript in the possession of the author, 4711 Davenport St., N. W., Washington 16, D. C.

The present table consists of values to 205D of $\pi^n/n!$, for $n = 1(1)160$. The first 110 entries were calculated from my table[1] of $\pi^{\pm n}$, $n = 1(1)110$, 205S, at least, using appropriate data from Uhler's tables[2] of $n!$ and $1/n!$. Beyond $n = 110$ each entry was calculated from its predecessor, and every fifth number was calculated independently as a check.

The table as a whole was checked by computing therefrom 205D approximations to $\sin \pi$ and $\cos \pi$. The respective values found were -2×10^{-205} and $-1 - 2 \times 10^{-205}$. Other data immediately obtainable were 205D values for $e^{\pm\pi}$, $\sinh \pi$, and $\cosh \pi$. The product of the approximations to e^π and $e^{-\pi}$ was found to equal $1 - 7.036 \times 10^{-205}$, nearly, indicating an additive correction to the calculated value of $e^{-\pi}$ of about 3.04×10^{-206}, which was subsequently confirmed by a second calculation of $e^{\pm\pi}$ using data carried to about 210D. Comparison of my values of these constants was made with the corresponding data published to 138S by Uhler,[3] and complete agreement to that degree of accuracy was found.

Then the tabular entries were collated with Hayashi's table[4] of $\pi^n/n!$, $n = 1(1)16$, 14–40D, and one error was detected therein, namely, in the 20th decimal place of π. It should be mentioned that Hayashi's decimals are not rounded. Six terminal digit errors ranging in magnitude from one to three units, and corresponding to $n = 6, 10, 11, 12, 13$, and 14, were discovered in the companion table of π^n.

On the basis of the present table there were computed tables of $(\pi/2)^n/n!$ and $(\pi/4)^n/n!$ to 40D, which were used to test the accuracy of smaller tables of ANDOYER,[5] PETERS & STEIN,[6] and BRANDENBURG.[7] No errors were detected in the first two mentioned; the errors in the last have already been noted by the writer in *MTAC*, v. 2, p. 46–47, MTE 71.

JOHN W. WRENCH, JR.

[1] *MTAC*, v. 1, p. 452, UMT 38.
[2] H. S. UHLER, (a) *Exact Values of the first 200 Factorials*. New Haven, 1944. (b) "A new table of reciprocals of factorials and some derived numbers," Conn. Acad. Arts and Sci., *Trans.*, v. 32, 1937, p. 381–434.
[3] H. S. UHLER, "Special values of $e^{k\pi}$, cosh $(k\pi)$, and sinh $(k\pi)$ to 136 figures," Nat. Acad. Sci., *Proc.*, v. 33, 1947, p. 34–41.
[4] K. HAYASHI, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*. Berlin, 1930.
[5] M. H. ANDOYER, *Nouvelles Tables Trigonométriques Fondamentales* (*Valeurs naturelles*). Paris, v. 1, 1915.
[6] J. T. PETERS & J. STEIN, *Zehnstellige Logarithmentafel*, v. 1, *Anhang*, p. 95. Berlin, 1922.
[7] H. BRANDENBURG, (a) *Siebenstellige trigonometrische Tafel*. Second ed., Leipzig, 1931; (b) *Sechsstellige trigonometrische Tafel*. Ann Arbor, Mich., 1944.

64[F].—R. J. PORTER, *Table giving the complete classification of primitive binary quadratic forms for negative determinants from $-D = 1001$ to $-D = 2000$*. Ms. computed from April, 1946 to August, 1947, the property of the author, residing at 266 Pickering Road, Hull, England. Compare *MTAC*, v. 1, p. 451.

The Ms. differs slightly in form from that described in *MTAC*, v. 2, p. 183–184, for $-D = 2$ to $-D = 1000$. In tabulating the periods, only the first half of each is given; the remaining half, comprising forms opposite to those in the first half, can be supplied by inspection.

The following results were obtained, including those already given in the article quoted above. The total number of forms is 44141. Of these, 3391 are given by 161 determinants of only one genus each, in which the longest period (of 69 forms) is afforded by the determinants -1831 and -1979; 15550 forms are given by 729 determinants of two genera each (of which, for the determinant -1889, each contains 36 forms); 19272 forms are afforded by 859 determinants each of four genera, the most frequent type being that where each genus contains six forms—this happens in 140 cases; 5768 forms are given by 243 determinants with eight genera each, none containing more than seven forms; and, finally, there are 160 forms given by 7 determinants (-840, -1320, -1365, -1560, -1680, -1785, and -1848) which yield sixteen genera (none of more than two forms each).

There are 53 irregular determinants (of which 18 in the first thousand have been previously given). The other 35 are -1075, -1088, -1107, -1187, -1220, -1228, -1259, -1267, -1312, -1315, -1323, -1332, -1356, -1440, -1508, -1513, -1539, -1568, -1582, -1590, -1598, -1600, -1675, -1701, -1725, -1755, -1763, -1764, -1780, -1836, -1872, -1886, -1918, -1931, -1971. The exponents of irregularity are either 2 or 3, 25 of the former and 28 of the latter.

R. J. PORTER

65[U].—NBSCL, *Table of Conversion Angles*. Manuscript prepared in 1947 to be published by the U. S. Hydrographic Office, Washington, D. C. 19 sheets, 28 × 37.5 cm.

This is a table of differences to the nearest tenth of a degree between the Rhumb Line Course and the Great Circle Course (assuming the earth to be a perfect sphere) for latitude of departure = 0(5°)85°, latitude of destination = 0(5°)90°—either in the same or in the opposite hemisphere of the point of departure. Difference between longitudes = 0(5°)120°.

Table 1, p. 15–16, *American Practical Navigator* (originally by N. BOWDITCH), U. S. Hydrographic Office, no. 9, revised edition of 1938, is for Radio Bearing Conversion. For difference of longitude = 1°(0°.5)16°.5 and middle latitude 4°(1°)60° the table gives the correction to be applied to radio bearing to convert to Mercator bearing. This table is to be replaced by the much more extensive new table.

With the new table

(1) Great circle directions can easily be converted to rhumb line directions for plotting radio bearing on a Mercator chart. Since radio waves travel along great circles, such corrections are necessary.

(2) Rhumb lines may be converted to great circle directions. Usually tangents or chords of the great circle, for about 5° of longitude, are sailed. The Table will be of value in planning courses, and will quickly show when the difference between rhumb line courses and great circle courses is significant.

If values are desired within the 5° intervals of the table at higher latitudes they may readily be found by double interpolation. The table may be of service in both marine and air navigation.

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AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

The leading article of this issue of *MTAC*, "A Bell Telephone Laboratories' Computing Machine—I," by Dr. FRANZ L. ALT, is our current contribution under this heading.

DISCUSSIONS

Decimal Point Location in Computing Machines

General. For various reasons, the majority of large scale automatic-sequence digital computers in existence or in the design stage use fixed decimal (or binary) point numbers. In all the machines of which the writer has knowledge, the decimal (or binary) point is located at the extreme left so as to make all numbers fall in the range -1 to $+1$. It is the writer's opinion that this choice is not the best, and that location of the decimal (or binary) point several digits away from the extreme left is better in almost every respect. Some of the considerations leading to this opinion are presented in this paper. For the sake of simplicity and definiteness, unless otherwise stated, the discussion refers to a machine in which data are stored in some form of memory which restricts numbers to a specified fixed number of decimal digits. This machine is assumed capable of automatically following a prescribed sequence of instructions which include addition, subtraction, multiplication, division, and transfer operations. The results of the operations are rounded off on the right to the same number of digits as the original data, after positioning the digits in accordance with a single fixed decimal point location. It is further assumed that the machine has no provision for handling excess digits on the left (exceeding capacity of the machine) created by any operation, so that such an occurrence implies an error in programming.

Possibility of Exceeding Capacity. One of the chief advantages claimed for the extreme-left-hand position of the decimal point is that the machine capacity as regards size of individual numbers cannot be exceeded in multiplication, since a product of two numbers, each less than unity, must also be less than unity. As this is not true for the other operations, such as addition or division, this feature would not appear to have much value, since magnitudes of numbers at each step have to be studied by the programmer in any case to avoid exceeding capacity in other operations. Even in multiplication, magnitudes must be considered to correct for the opposite effect caused by this choice of decimal position (the continual decrease in magnitudes of numbers as a result of successive multiplications) by judicious shifting (or division by powers of 10) at suitable stages in the computation.

Although any choice of the decimal point further to the right creates the possibility of obtaining products larger than either factor, thus exceeding limits if the factors are sufficiently large, this is compensated somewhat by the possibility of arranging values (for example by keeping them approximately equal to unity) so that any number of successive multiplications may be programmed without either exceeding limits or losing many significant digits. Furthermore, in division, use of wholly fractional numbers requires that the numerator be smaller than the denominator to keep the quotient within bounds, while much broader limits are permissible if one or more digits are available to the left of the decimal point. The extra magnitude of the checking in division necessitated by choice of the extreme-left decimal point would appear to offset any possible gain in checking of multiplications.

Inherent Accuracy. Another important advantage claimed for use of the extreme-left decimal point location is that this permits maintenance of the greatest number of significant figures in each factor in multiplication and hence leads to greatest accuracy. That this contention cannot be wholly true is easily seen from the following 3-digit example, which compares the result obtained when the decimal point is at the extreme left with that obtained when the decimal point is one digit to the right.

$$.123 \times .111 = .014 \qquad \text{accuracy 2 digits}$$
$$1.23 \times 1.11 = 1.37 \qquad \text{accuracy 3 digits.}$$

Let us examine the question of accuracy in a little more detail. We restrict ourselves to consideration of errors due to the limited number of digits (round-off errors) only and, for simplicity, evaluate the maximum such error instead of the most probable value. We further assume that the programmer has accurate cognizance of the magnitudes of all numbers and can shift them to the most advantageous digital position, subject to the number limitations of the machine, but that no auxiliary operations such as division or multiplication by factors other than powers of 10 are to be employed, so that there is no change in the digits themselves. If numbers have n total digits and the decimal point is m digits from the left, the maximum round-off error in a number of magnitude N is $10^m K/N$ of the number, where

$$K = \tfrac{1}{2} \times 10^{-n}$$

is the error for maximum N. If two numbers M and N are multiplied together, their product will be $P = MN$. The percentage error in P is the

sum of the percentage errors in M and N plus additional round-off error in limiting P to the specified number of digits, making the maximum error in P equal to

$$E = 10^m(1/M + 1/N + 1/P)K,$$

where all values are assumed positive.

For a machine with decimal point at the left, $m = 0$ and M, N, and P must not exceed unity. Minimum E occurs for numbers close to this limit and has the value

$$E \text{ (minimum for } m = 0) = 3K.$$

Maximum E occurs when M and N (and hence P) are smallest. Since, under our assumptions, we may always shift numbers so that the first digit is not a zero, we can insure that M and N are at least .1. For this limit, we have $P = .01$ so that

$$E \text{ (maximum for } m = 0) = (10 + 10 + 100)K = 120K.$$

For a machine with decimal point one digit to the right, $m = 1$, and M, N, and P must not exceed 10. Minimum E occurs for $M = N = \sqrt{10}$ and has the value

$$E \text{ (minimum for } m = 1) = 10(1/\sqrt{10} + 1/\sqrt{10} + 1/10)K = 7.3K.$$

Maximum E occurs when M and N (and hence P) are smallest. Whenever possible, we shift M and N so that the first digit in each is not a zero. When this causes P to exceed 10, we shift the number (say M) with smaller initial digits to the extreme left and allow a single zero digit at the left of the other. With this arrangement, which is always possible, the largest E occurs for $M = \sqrt{10}$, $N = 1/\sqrt{10}$, $P = 1$ and is

$$E \text{ (maximum for } m = 1) = 10(1/\sqrt{10} + \sqrt{10} + 1)K = 45K.$$

We thus see that the maximum error is actually less in this case by a factor of almost 3 as compared to the case where the decimal point is at the extreme left, although the minimum error is not as low. Even for $m = 2$, we get

$$E \text{ (maximum for } m = 2) = 120K,$$

corresponding to M close to 10 and $N = 1$, so that the extreme error, with the decimal point two digits to the right, is no greater than with decimal point at the extreme left.

The above is based on the assumption that the programmer can predict magnitudes very closely and that, in iterated operations, the range of magnitude variation is small. This is generally not the case, and an appreciable spread of values must be allowed for. If we assume uncertainty in each of the factors in a multiplication over a range of 10 to 1, the programmer cannot assure best placement of digits but only that numbers will be at most one digit away from best positioning. Maximum errors will then occur for M and N both one tenth their previous values. For $m = 0$, this corresponds to $M = N = .01$; $P = .0001$, giving

$$E \text{ (} m = 0, \text{ 10 to 1 range)} = (100 + 100 + 10000)K = 10200K.$$

For $m = 1$, the extreme occurs when $M = 1/\sqrt{10}$; $N = 1/10\sqrt{10}$; $P = 0.01$ and is

$$E\ (m = 1,\ 10 \text{ to } 1 \text{ range}) = 10(\sqrt{10} + 10\sqrt{10} + 100)K = 1350K,$$

being almost 10 times as good as the preceding. Similarly, for $m = 2$, $M = 1$; $N = 0.1$; $P = 0.1$ is the extreme case and

$$E\ (m = 2,\ 10 \text{ to } 1 \text{ range}) = 100(1 + 10 + 10) = 2100K.$$

Even for $m = 4$, the extreme is for $M = 10$; $N = 1$, $P = 10$ and has the value

$$E\ (m = 4,\ 10 \text{ to } 1 \text{ range}) = 10000(.1 + 1 + .1) = 12000K,$$

which is not materially greater than for $m = 0$.

Obviously, the greater the range of values in an iterated operation, or the greater the uncertainty in regard to possible magnitudes that may occur, the poorer the choice of decimal point at the extreme left becomes as concerns accuracy, and the greater the number of digits to the left of the decimal point corresponding to minimum extreme error.

Availability of Common Integers and Constants Exceeding Unity. It is certainly highly desirable to have integers such as 1, 2, 3, or 10 and constants such as $\pi = 3.14159$, $e = 2.71828$, $\sqrt{2} = 1.41424$, $\ln 10 = 2.30259$, etc., available for use in programming. With the decimal point at the extreme left, such values cannot be expressed directly but must be divided by a power of 10 or must be given in a different form such as the reciprocal value. This, at best, complicates the programmer's work. As a simple example, consider the usual square root iteration

$$y_{n+1} = (y_n + x/y_n)/2$$

to obtain $y = \sqrt{x}$. The convenient first approximation of $y_0 = 1$ cannot be directly used, nor can the operation be carried out as indicated. It is necessary to change the form to

$$y_{n+1} = \tfrac{1}{2}y_n + \tfrac{1}{2}x/y_n$$

and to take a string of 9's for y_0 to avoid exceeding capacity. This involves one additional initial step (halving of x) and an awkward value for y_0 and still is insufficient if there is any possibility that x itself is a string of 9's, since this would cause y_1 to exceed unity (because of normal round-off's).

Mathematical values do not naturally limit themselves to magnitudes below unity, and such an arbitrary limitation may be expected to lead to various programming difficulties, which may be minimized by allowing some range above unity. The most important value that should be included in the permissible range is unity itself. If, for example, cosine or sine values are required in a computation, it is possible to exceed the capacity of a machine with decimal point at the left if the argument should get too close to zero or $\frac{1}{2}\pi$, requiring either some assurance that this will not occur or sacrifice of a decimal digit for values with the accompanying difficulty of introducing additional factors in the programming. It is certainly far more convenient for the programmer to work with natural values than to keep track of various factors introduced to shift values to lie within range of the machine. Of course, some factors are required with any choice of decimal point loca-

tion, but their number is materially greater for the extreme left location than for locations several digits further to the right. It is only necessary to glance through a mathematical text to verify this statement by observing the relative occurrence of values above, say, 100 as compared to those between 1 and 100.

Problems Involving Extreme Accuracy. A general-purpose computer may be required to solve problems whose solution is desired to greater accuracy than corresponds to the fixed number of digits permitted in a single memory location of the machine. In certain problems, while the solution is not required to excessive accuracy, it is necessary at some intermediate stage to provide much greater precision because of large loss of relative accuracy during the process of computation. Before discussing this matter in regard to decimal machines, it may be noted that, in addition to difficulties similar to those encountered in decimal machines, a binary machine used for problems of extreme accuracy involves questions of precision of conversion between the normally used decimal system and the binary system used by the machine. Since, in general, only integral values convert exactly between these two number systems, a binary machine with binary point at the extreme left requires the programmer to insert powers of 2 as factors as well as powers of 10 in order to avoid conversion errors. For example, 10 would be inserted as $\frac{5}{8}$ with the factor 16 to be remembered by the programmer. While such binary factors can probably be handled expeditiously by high-grade mathematicians, they are certainly more likely to cause errors and require special training for programmers of lower skill.

In order to simplify handling of high-precision problems, we will assume that our machine has available, when called for, a special division process which, in addition to the normal quotient, stores the remainder (shifted to the left up to the decimal point) into a specified memory location. Such a facility as well as a similar one for multiplication will be available in the EDVAC now under construction by the University of Pennsylvania. For a machine of 3 decimal digits with decimal point at the left, this special division will work as follows:

$$.111/.123 = .902 + .000054/.123.$$

The machine will store .902 in one location and .054 in another. For a similar machine with one digit to the left of the decimal point, this operation will be:

$$1.11/1.23 = 0.90 + .0030/1.23,$$

and the machine will store 0.90 in one location and 0.30 in another. In all cases, the second number will always be less than the divisor.

With the aid of this special operation and a corresponding multiplication, and more than one memory location for the results of each operation, almost any degree of accuracy can be achieved. It is to be noted, however, that each value is expressed as a sum of component terms, with suitable factors, each stored as a separate number. Successive operations are, therefore, operations on polynomials and result in similar polynomials. At each stage, care must be taken not to exceed capacity in any term of the polynomial and to "carry over" to the next term any indicated excess. With the decimal point at the extreme left, this is a difficult chore and generally requires

extensive modification of procedure with addition of quite a few comparison instructions (branch orders) to check for and correct this condition without momentarily exceeding capacity at some point. With one or more digits to the left of the decimal point, this problem becomes much simpler to take care of since, as will be noted in later numerical examples, all terms other than the first are normally less than unity. It is thus possible to perform the next operation without considering the exceeding of capacity as a result of it, and then to add the integral portion of each term to the preceding (after suitable shift to the right) as a "carry." In fact, this "carry" process may frequently be postponed until after a number of operations are carried out. For example, the evaluation of e to any number of digits can be completely carried out on a machine of the type we are discussing, with one digit to the left of the decimal point, with only a single "carry" routine in the entire procedure.

It may be worth while to give a simpler numerical example here. Suppose we had a 3-digit machine and wished to evaluate

$$101/301 + 201/401$$

to 7 digits. Consider first the decimal point at the extreme left. Using special division, we obtain

$$\begin{aligned}
.101/.301 &= .335 \text{ quotient} + .165 \times 10^{-3} \text{ remainder}\\
.165/.301 &= .548 \text{ quotient} + .052 \times 10^{-3} \text{ remainder}\\
.052/.301 &= .173 \text{ normal quotient}
\end{aligned}$$

so that we store 101/301 in 3 memory locations as

$$.335 + .548 \times 10^{-3} + .173 \times 10^{-6}.$$

Similarly, we get

$$201/401 = .501 + .246 \times 10^{-3} + .883 \times 10^{-6}.$$

We cannot now add terms directly since their sums may exceed unity. It is thus necessary to examine values before summing them. One way of doing this is to compare the complement of .173, which is .827, with .883. Since it is smaller than the latter, we subtract it from the latter obtaining .056 and carry 1 to the next term. Comparing the complement of .548, which is .452, with .246, we find it larger, indicating no carry. We thus add the original terms, obtaining .794, to which we must add .001 carry from the previous addition, requiring another comparison to show that we may directly add to get .795 with no carry. The first terms can, of course, be directly added without any ado since we know their order of magnitude. We thus get

$$101/301 + 201/401 = .836 + .795 \times 10^{-3} + .056 \times 10^{-6}.$$

If additional similar terms were to be added, the work would, of course, increase enormously.

Now let us consider the same problem with the decimal point one digit to the right. As before, we get

$$\begin{aligned}
1.01/3.01 &= 0.33 \text{ quotient} + 1.50 \times 10^{-2} \text{ remainder}\\
1.50/3.01 &= 0.55 \text{ quotient} + 1.45 \times 10^{-2} \text{ remainder}\\
1.45/3.01 &= 0.48 \text{ quotient} + 0.52 \times 10^{-2} \text{ remainder}\\
0.52/3.01 &= 0.17 \text{ normal quotient}
\end{aligned}$$

so that we store 101/301 in 4 memory locations as

$$0.33 + 0.55 \times 10^{-2} + 0.48 \times 10^{-4} + 0.17 \times 10^{-6}.$$

Similarly, we get

$$201/401 = 0.50 + 0.12 \times 10^{-2} + 0.46 \times 10^{-4} + 0.88 \times 10^{-6}.$$

Here we may add corresponding terms without worry of exceeding capacity. In fact, since each term is less than unity in value and the machine capacity is 10, we could add 10 terms at a time without fear of exceeding limits. We thus get

$$101/301 + 201/401 = 0.83 + 0.67 \times 10^{-2} + 0.94 \times 10^{-4} + 1.05 \times 10^{-6}.$$

We now take the integral part of each term as a "carry" to the preceding term. If less than 9 terms had been previously added, such carry cannot cause excessive magnitudes and can be very simply carried out in succession. In this case, only one carry occurs and we finally get

$$101/301 + 201/401 = 0.83 + 0.67 \times 10^{-2} + 0.95 \times 10^{-4} + 0.05 \times 10^{-6}.$$

The only disadvantage of the new decimal point location is that it requires more terms for equal accuracy; however, the given example accentuates this difficulty because of the small number of total digits and the particular choice of numerical values. For usual machines of 10 or more decimal digits, the increase in number of terms caused by one digit shift of the decimal point would be negligible, but the saving in carry-over difficulties just as large. Choice of numerical values such as 41/301 + 51/401 would further equalize the number of terms.

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BIBLIOGRAPHY, Z-II

1. ARTHUR W. BURKS, HERMAN H. GOLDSTINE & JOHN VON NEUMANN, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument.* Prepared for the Research and Development Service, Ordnance Department, U. S. Army, June, 1946, i, 53 leaves and table, 21.6 × 27.9 cm. Mimeographed.

This report, consisting of 6 sections and a table giving a proposed instruction code, is the first of two papers dealing with "some aspects of the over-all logical considerations arising in connection with electronic computing machines."

Section 1.0, "Principal Components of the Machine," is a statement of the "main organs" a fully automatic general-purpose computing machine must possess. The authors point out that such a machine must have a memory organ, a control organ, an arithmetic organ, and input and output organs.

The computer must be capable of storing tables of functions, the original data of problems, and the intermediate results of computations; it must also be able to store sequences of instructions for performing specified numerical computations. This storage function is performed by a memory organ. The

automatic execution of the orders stored in the memory is in turn effected by the control organ.

A general-purpose calculator must be capable of performing the arithmetic operations. The machine will view certain of these operations as elementary and will have an organ—the arithmetic organ—to perform them automatically upon demand. This organ will add, subtract, multiply, divide and perform other operations such as shifting, frequently occurring. Lastly, the computing machine must possess input and output organs to serve as communication channels between it and the human operator.

Section 2.0, "First Remarks on the Memory," is concerned with the size of the memory organ. Here are discussed briefly the memory requirements of various types of computations, including the numerical solution of total differential equations, of partial differential equations of the parabolic or hyperbolic type, and of problems solved by iterative procedures, such as systems of linear equations and elliptic partial differential equations. It is concluded that an electronic memory organ capacity of 4000 numbers of 40 binary digits each exceeds the capacities required for problems handled by existing machines by a factor of about 10. The machine proposed in the report will have an electronic memory capacity of this size and will have, in addition, a subsidiary memory, also fully automatic, of larger capacity but slower in the transfer of numbers and orders.

In Section 3.0, "First Remarks on the Control and Code," a type of instruction code envisaged by the authors is analyzed. A code must be adequate for the description of any sequence of operations the computing machine may be expected to perform. Decisive considerations in the choice of a code are the simplicity of the equipment it requires, the ease with which the human operator can use it to instruct the machine to work important problems, and the speed with which the machine can interpret and follow it.

Minimum requirements for a code are given. The code must contain instructions for performing the fundamental arithmetic operations, for transfer of data from the memory to the arithmetic organ and back again, and for the modification of memory location numbers of orders. Orders which will integrate the input-output devices with the machine will also be necessary. Finally, since the control organ will not always follow the normal sequence from place n in the memory to place $(n + 1)$ for its next instruction, the code must include control transfer orders.

The ideal memory unit, described in Section 4.0, "The Memory Organ," would be capable of storing an indefinitely large number of 40-binary digit numbers or words in such manner that each word would be available within a few micro-seconds. It is stated that it appears physically possible to achieve indefinitely large memory capacity only by the use of a hierarchy of memories, each of which has greater capacity than the preceding but which is less quickly accessible for the transfer of stored information. A possible hierarchy would consist of a primary high-speed memory unit of relatively small capacity, a secondary memory, slower but of larger capacity, and a vast quantity of dead storage, i.e., storage not integrated with the computing machine.

A suitable primary or high-speed memory device appears to be a cathode-ray tube storing electrical charges on a dielectric plate inside the tube. Such a tube is effectively a myriad of electrical capacitors which can be connected

into the memory circuit by means of an electron beam. The Princeton Laboratories of the Radio Corporation of America are engaged in the development of the Selectron, a storage tube of this type. For the high-speed memory it is planned to use 40 Selectrons in parallel, each tube storing one digit of a 40-binary digit number. Such parallel use of Selectrons provides a faster memory and appears to involve simpler techniques than the use of Selectrons in series. The parallel-type memory causes the computer to be essentially of the parallel type; for example, in performing an addition all corresponding pairs of digits are added simultaneously. (In a serial computer, these pairs of digits are added serially in time.) In fact, the authors point out that once a given component has been chosen as the elementary memory unit, the nature of the balance of the computing machine is more or less determined.

The secondary memory can consist of storage on teletype tapes, magnetic wire or tapes, movie film, or similar media controlled by the Control Organ. The automatic integration of this storage medium with the computer is achieved by introducing appropriate orders into the code. The secondary storage medium must be such that the operator can place words on the medium and can read words put on it by the computing machine. It serves as a part of the input-output system of the machine. The dead storage is an extension of the secondary storage medium, differing only in not being immediately available to the computing machine. It may consist of a library of tapes that can be introduced into the machine by the human operator when desired and which will then be automatically controlled by the computer.

Section 5.0, "The Arithmetic Organ," includes a discussion of the features the authors consider desirable for the arithmetic part of a computer. Tentative conclusions are reached as to which of the arithmetic operations should be built into the machine. The remaining operations must be programmed. A schematic drawing of the arithmetic unit is described.

The planning of the arithmetic organs of a computing machine leads naturally to a consideration of the number system to be adopted, to a determination of whether the machines shall store numbers in terms of significant figures or decimals (i.e., "floating" versus fixed decimal point), to a study and evaluation of rounding-off procedures, and to the analysis of iterative processes that can be used as substitutes for built-in arithmetic operations. These matters are treated clearly and in detail (21 leaves of a 53 leaf report). Perhaps the most interesting characteristic of this section is the manner in which arithmetical and statistical analysis is correlated with the design of the arithmetic organ. The proposed computing machine is described as a 40-binary digit, fixed-binary-point machine with a double precision multiplier, in which addition, subtraction, multiplication, division, shifting, and taking the absolute value are "built-in" arithmetic operations.

Section 6.0, "The Control," describes the function of the Control Organ. An order code for the operations to be performed within the computer is proposed and given in Table I at the end of the section. Orders relating to operations involved in getting data in and out of the computer are not discussed. The control routine in the execution of the built-in orders is analyzed and the general manner in which the control organ functions is described. It is indicated how decoding or many one-function tables serve as switches

by means of which the control selects a specified Selectron memory location and decodes the orders, which are given in coded binary form. How binary counters and flip-flops (trigger circuits) enable the control to take order pairs in sequence from the memory and shift itself from one sequence of control orders to another is described briefly.

The merits of various alternative checking systems are weighed against their cost in equipment. The method of localizing errors is discussed. The proposed computer has a feature the authors consider advantageous in the light of their experience with the ENIAC. Namely, the circuits are designed in such a manner that if the timing clock is stopped between pulses, the computer will retain all its information in trigger circuits so that the computation may proceed unaltered when the clock is started again. The operator will then be able to put the machine through an operation step by step, checking the results by means of the indicating lamps connected with the flip-flops. It is clear that this will be an extremely useful feature of the computer.

The report concludes with a discussion of the function of the control organ in the execution of input-output orders. It is apparent that the authors' views on input-output organs are not as definite as those on the other organs described in the report.

This report presents a clear picture of the factors that were taken into account by a competent group in planning an automatic-sequence electronic digital computing machine, and is thus interesting and useful to anyone who is active in the large-scale electronic computer field. Its main value lies in the fact that the authors have succeeded in giving a general picture of the type of computer under consideration, and in providing a background of the logical considerations underlying their decisions concerning the kind of computer that would be most useful for general mathematical computation.

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2. L. J. Comrie, "Calculating—past, present and future," *Future, the Magazine of Industry, Government, Science, Arts*, 1947, no. 1, "Overseas Issue," p. 61–69, 15 illustrations. 20.6 × 29.5 cm. Published in England but printed in Holland.

In this popular article, Dr. Comrie presents an interesting survey of the development of calculating machines from the use of small stones as counters up to the modern ENIAC built at the Moore School of Engineering, University of Pennsylvania. He shows the progression from the early, primarily commercial machines to those designed and built by scientists for the solution by numerical evaluation of equations classified as insoluble by the pure mathematician. The conception of the ENIAC as an electronic brain—a false impression that seems to have captured the public imagination—is swept aside by the author. He considers the machine in no way a substitute for the human mind, which must still instigate the machine's every move; the rapid calculation of problems heretofore impracticable of solution is seen as its chief function.

Charles Babbage's "analytical engine" is described as one of the most important forerunners of the modern calculators. This machine, with a store for numbers, a mill where calculations were to be performed, and a

control to link these two units, was never completed because of production difficulties and lack of financial support. However, it has served as a stimulus to the development of the present IBM Sequence-Controlled Calculator, the Bell Telephone Relay Machine, and the ENIAC. It is pointed out that the development of the Differential Analyzer by Dr. Bush at the Massachusetts Institute of Technology gave further impetus to the new trend in calculating, which emphasizes speed and the elimination, as far as possible, of human intervention.

The Harvard IBM machine, "the first of the really big machines brought into being by the war to produce actual numerical results," consists largely of specially assembled Hollerith parts and has as its "brain" a newly designed tape control. Professor H. H. Aiken, who collaborated with IBM engineers in the creation of this machine is now at work on an improved calculator of this type. Another war-time development was the Bell Telephone Laboratories Relay Machine, designed by Stibitz & Williams. It is organized on very much the same basis as a desk calculator and bears a resemblance to a telephone exchange. The ENIAC, designed to meet the needs of the U. S. Army Ballistics Research Laboratory at Aberdeen, Maryland, attains much greater speed by the use of electronic instead of mechanical parts. Although it is much too large and contains far too little memory space, it has served to pave the way for the current EDVAC machines now being designed at Philadelphia and Princeton, and for the general-purpose British ACE, also in a design stage.

This entertaining article, containing more than a dozen excellent illustrations of historical as well as modern machines, emphasizes the pressing need for continued support of the development of the new calculators.

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3. Herman H. Goldstine, and John von Neumann, *Planning and Coding of Problems for an Electronic Computing Instrument*, Institute for Advanced Study, Princeton, New Jersey, 1947, 69 pages, 21.6 × 27.9 cm.

To the steadily increasing group of engineers and mathematicians who are becoming electronic-machine conscious, two reports prepared by the Institute for Advanced Study for the Army Ordnance Department have proved of invaluable aid. The first,[1] which outlined the features of a high-speed computing machine possessing a large selectron-tube memory, carried a table of 21 orders, or codes,[2] of such a nature that the ability of the machine to execute them would render it a powerful tool for solving highly complex mathematical problems. This table, with several modifications, is repeated in the second report—the volume under review—which constitutes a manual on the application of these codes to actual problems.

When first faced with the task of setting up a problem on a high-speed machine, the coder's situation is like that of a navigator in unfamiliar waters. A route must be charted consisting of a series of steps for the control of the machine to follow from the starting point to the successful solution of the problem. Unfortunately, these steps (except in the case of the simplest problems) do not lie in a straight, unbroken, one-way path, but usually meander through a tortuous maze of loops, bifurcations, retracings, and jumps, as the computations may entail any combination of iterative proc-

esses, alternative procedures, and various modifications conditioned by intermediate results obtained during the process.

In the first section of their manual, the authors have laid down the general principles for the construction of such charts, or "flow diagrams," with amazing skill and inventiveness. They list and carefully analyze the various turns and twists that the future pilot-coder is likely to encounter and point out the location of dangerous reefs he must avoid. In order to designate on the flow chart all possible digressions from the straight path that may occur in the course of a problem's solution, they invent a set of comprehensive signals to be attached at strategic points.

The authors then proceed to illustrate these principles by application to actual problems. The first two of the five examples coded in the second section are too simple to require flow diagrams. They serve, however, to introduce the reader to several elementary coding principles, thus enabling him to follow the more difficult situations exhibited by the succeeding problems, each of which is accompanied by its flow diagram.

The flow diagram for Problem 3, which calls for the computation of the values of a quotient of two polynomials in u for a set of given values of u, has a loop that must be traversed repeatedly until all the required divisions are performed. Problem 4 entails a much more formidable flow diagram. The machine under consideration in this report computes with binary numbers of absolute value less than unity. It is essential that the machine be instructed to determine the magnitude of the quotient of two such numbers when it occurs, so that the proper scaling factor may be introduced to render the quotient suitable for further manipulation. Problem 4 illustrates how such automatic sensing of size can be coded. By the time the reader has covered the various steps contained in Problem 4, he is prepared to follow without difficulty the diagram and coding of the last problem in the section, that of finding the square root of a number by the process of iteration.

The third, and last, section of the manual deals with various systems of digital notation and discusses the possible input and output speeds of data. These are related to the projected speed of binary-decimal conversion within the machine. There follow two problems concerning binary-to-decimal and decimal-to-binary conversion, although it would seem that the latter operation could well be delegated to several subsidiary converters. The section ends with two problems illustrating the coding of double-precision arithmetic processes. These problems show in detail how the machine may be instructed to add, subtract, and multiply two numbers containing 78 binary digits each, instead of the ordinary 39 digits. A brief outline for coding double-precision division is also given. It is pointed out that double precision will take 6 to 8 times longer than ordinary precision operations.

As in the case of the earlier manual[3] issued by the Computation Laboratory at Harvard, the present manual is of greatest benefit in so far as it contains principles and illustrations sufficiently general to apply to any high-speed sequence machines about to be constructed. It does not detract from the great value of these reports that the codes contained in them may perhaps be revised in character and number to simplify either the construction of the machine, the task of the coder, or both.

The authors themselves admit the lack of finality in their coding system.

What they attempt to do in their report is to indicate that (to quote their own words) "coding . . . has to be viewed as a logical problem and one that represents a new branch of formal logics." In this they succeed admirably, as well as in formulating the laws of this new branch. The series of forthcoming reports on the same subject, which are promised the reader in the Preface, will be eagerly awaited.

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[1] A. W. BURKS, H. H. GOLDSTINE, & J. VON NEUMANN, *Preliminary Discussion of the Logical Design of An Electronic Computing Instrument*, see no. 1 in this Bibliography.
[2] These do not include the necessary input-output orders.
[3] HARVARD UNIV., Computation Laboratory, *A Manual of Operation for the Automatic Sequence Controlled Calculator*, 1946, see *MTAC*, v. 2, p. 185–187.

4. D. R. HARTREE, "The application of the differential analyzer to the evaluation of solutions of partial differential equations," *Proceedings of the First Canadian Mathematical Congress, Montreal, 1945*. Toronto, Univ. of Toronto Press, 1946, p. 327–337. 14.6 × 20.96 cm.

In this short technical paper, Professor Hartree first mentions the possibility of a quantitative solution on the differential analyzer of certain partial differential equations of applied mathematics that are inconvenient or impossible to solve in terms of tabulated functions. He then briefly describes this machine, which was developed by Dr. Vannevar Bush at the Massachusetts Institute of Technology primarily to solve ordinary differential equations by mechanical means. Its units, which may be interconnected by shafts and gearing in many different ways depending on the specific problem, translate the required mathematical operations into mechanical terms. Chief among these units are integrators capable of performing integrations continuously with respect to any variable, adding units, input and output tables, and in some cases a multiplier. The machine is used most effectively, says the author, when a number of solutions of a single equation or a solution of a series of equations of the same type is required, all of which would necessitate comparatively minor changes in the machine setup.

More specifically, Professor Hartree discusses the application of this machine to the solution of a two-point boundary problem in which an estimate of some quantity must be made at the beginning of the integration and trial solutions worked through until the required condition at the other end of the range of integration is satisfied. In the case of simple heat conduction in one dimension in a slab of a uniform solid whose thermal properties are constant, a solution is obtained by replacing the original partial differential equation with an approximately equivalent ordinary differential equation. This may be done by working in finite differences in the time variable and integrating continuously in the temperature distribution variable, although several hundred trial runs are necessary before an accurate solution is thus reached. Alternatively, the author outlines a more recent method that is less time-consuming but more demanding on the machine. It consists essentially in integrating in time and replacing the space derivative by finite differences. This is effected by repeatedly selecting three equally spaced points in the field and replacing the original heat equation with a finite difference equation involving these points. A simultaneous solution of the resulting equations is therefore necessary but may be accomplished in only

one machine run. Either of these methods is applicable, as the field of integration in this particular problem is open in the time direction. If the field of integration were surrounded by a closed boundary, it would be more efficient to apply Southwell's relaxation method, which does not come within the scope of the machine.

Several extensions of the heat conduction equation, to which both of these methods might be applied, are also mentioned. A list of nineteen references is appended to the article.

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News

Eastern Association for Computing Machinery.—The first meeting of this group was held on September 15, 1947, at Columbia University, New York City. An interesting description of the pilot model of the EDVAC, illustrated with slides, was given by Dr. T. K. Sharpless of the Moore School of Electrical Engineering. The following officers were elected:

J. H. Curtiss of the National Bureau of Standards, President;
J. W. Mauchly of the Electronic Control Company, Vice-President;
Edmund C. Berkeley of the Prudential Insurance Company of America, Secretary;
R. V. D. Campbell of the Raytheon Manufacturing Company, Treasurer.

The officers were instructed by the committee of the whole to appoint eight additional members of an Executive Committee, which is to write a set of by-laws for the Association. The by-laws will be submitted for approval within three months to those interested in the formation of the Association.

The Institution of Electrical Engineers.—On March 7, 1947, a discussion was held on "Recent Developments in Calculating Machines" at the Institution of Electrical Engineers, Measurements Section, London, England. Professor D. R. Hartree opened the meeting with a brief mention of the most important analogue and digital machines, illustrating his remarks with photographs of the newest differential analyzer at M. I. T., of the two Harvard machines, and of the ENIAC. He also suggested improvements in the ENIAC to make it a general-purpose machine, namely, a larger memory and the elimination of manual programming. Dr. Turing further suggested that an adequate memory could be used to eliminate the need for any manual programming.

Mr. John Todd, now a member of the staff of the Institute of Numerical Analysis, although admitting the importance of the new machines, pointed out that the Admiralty Computing Service had been able to handle all the work submitted to it using only desk computers. (See *MTAC*, v. 2, p. 289f.) Great care should be exercised, said Mr. Todd, in the formulation and preparation of problems for computation.

In answer to Dr. Hartree's suggestions for a computing machine of a more general nature than the ENIAC, Dr. Comrie stated that the EDVAC, which has been declassified, does meet Professor Hartree's requirements. He also heartily endorsed the proposal of Todd and Erdélyi for an Institute of Practical Mathematics,[1] which would be of great value for the effective utilization of the projected machines.

After a short talk on iconoscope storage devices by Professor F. C. Williams, of Manchester, Professor Hartree, citing problems of quantum mechanics and of supersonic motion, pointed out that there would be no lack of work for the new machines. He remarked that many essentially nonlinear problems, heretofore impracticable, could be handled by these machines, necessitating new mathematical techniques and theories. A typical electronic machine, once the design problem had been overcome, would probably cost around £20,000. Professor Hartree admitted that the problem of checking had so far received too little attention. Running two machines in parallel would effectively eliminate any danger of machine error, but the more serious type of error, arising in the formulation of a problem, could only be avoided by more comprehensive checking.

Mathematical Association of America.—On September 2, 1947, at the concluding session of the summer meeting at Yale University, a symposium on computing machines was held, with lectures by HOWARD H. AIKEN of Harvard University and JOHN VON NEUMANN of the Institute for Advanced Study at Princeton University. Supplementing his lecture with slides, Dr. Aiken described the essential design features of the two electro-mechanical digital computing machines, Mark I and Mark II, which were built under his direction. He stated that the successful operation of the Mark I computer has proved the feasibility of big automatic calculators for large-scale computing, and that in the next few decades vast improvements may be expected in this field.

Dr. von Neumann, who is directing a research project at the Institute for Advanced Study for the construction of an improved electronic digital computing machine, spoke on some general aspects of the new high-speed computing field. In discussing the need for large-scale computing he stated that nearly all of the progress in solving the non-linear partial differential equations of aerodynamics so far has been brought about through the use of the wind tunnel, which may be regarded as a crude analogue calculator. He then spoke of the advantages of digital computers over analogue computers, explaining that the latter are completely inadequate to solve the difficult types of problems that digital computers could handle. He concluded with some speculations on the influence large-scale computers will have on future studies in pure mathematics, expressing his hope that the new machines will enable mathematicians to attack successfully the field of non-linear problems, where the classical methods of analysis, unaided by high-speed computing, have so far been unproductive.

Recent Developments in Mathematical Computing in France.—The Laboratoire de Calcul Mécanique, at Paris, is conducting a program of research in mathematical computing sponsored by the Centre National de la Recherche Scientifique (see *MTAC*, v. 2, p. 251). This project, which is under the direction of Dr. PIERRE LOUIS COUFFIGNAL, comprises two parts—one an investigation of computational methods, and the other the design and construction of an electronic digital computer. Although the design of the machine is not yet complete, certain guiding principles have been outlined, and the development of necessary components has been undertaken.

[1] A. ERDÉLYI & JOHN TODD, "Advanced instruction in practical mathematics," *Nature*, v. 158, Nov. 16, 1946, p. 690–692. Dr. Erdélyi is also now in this country, at California Institute of Technology, preparing papers of HARRY BATEMAN for possible publication. —EDITOR.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–II

1. GEORGES BAUDOUIN, "Principe d'une règle à calcul présentant une échelle logarithmique de grande longueur," Acad. d. Sci., Paris, *C.R.*, v. 224, Jan. 1947, p. 96–97.

Last paragraph: "Ainsi, avec une règle de dimensions restreintes, soit 20 cm. sur 4 cm., *on peut*, grâce au découpage de l'échelle logarithmique de 10 à 100 en dix parties superposées, *obtenir une précision environ dix fois plus grande qu'avec les règles ordinaires*." See *Math. Rev.*, v. 8, 1947, p. 289, E. LUKACS.

2. The GLOBE-HILSENRATH *Azimuth Computer. Instructions for the use of the Globe-Hilsenrath Azimuth Computer*. A. M. Messer & Co., 18 40th St., Irvington, N. J. 1945. 12 p. 12.7 × 17.7 cm. Instrument and Instructions $8.75. Distributor for School and College trade: Yoder Instruments Co., East Palestine, Ohio.

This computer consists of a white vinylite sheet, 23.3 × 24 cm., carrying on each of its faces a printed grid, and a transparent plastic protractor with movable radial arm which can

be attached in proper orientation to either side of the vinylite sheet. It is designed to be used in determining the azimuth of a celestial body from its local hour angle and declination and the latitude of the observer.

The grid on the face is intended to cover problems involving latitudes and declinations less than 45°, north or south. That on the back is for latitudes 40° to 60°, declinations 0° to 60°, north or south. Each grid consists of a set of confocal half-ellipses, to each of which corresponds a different degree of latitude and an orthogonal set of confocal hyperbolas, to each of which corresponds a local hour angle which is an integral multiple of four minutes. For local hour angles, 4 to 8 hours, and latitudes 20°S to 20°N, some of the curves are omitted. The declination scale is made up of the points where the latitude ellipses meet the line through their common center and normal to their major axes; for convenience's sake, however, this scale has been duplicated a short distance below and a red index line on the protractor allows one to set on the proper declination. The declinations of some of the more common navigation stars are marked on the declination scale, but because of the shortness of the declination scale, some, as Acrux, Rigil Centauri and Dubhe, often used by the navigator, are omitted.

To use the computer, one has only to mount the protractor on the proper side, set the protractor index on the declination of the celestial body, and turn the radial arm of the protractor until it passes through the intersection of the appropriate latitude ellipse and local-hour-angle hyperbola. One can then read Z_n, the azimuth of the celestial body directly on the protractor scale which is divided in degrees; one can estimate tenths of a degree if one wishes.

The most obvious criticism to be made of the computer is that the local hour angle is marked in hours and minutes instead of degrees and minutes of arc as is the custom today. Another weakness of the instrument is that there is no provision for the determination of azimuths for bodies of declination greater than 45° when the latitude is less than 45°; as a matter of fact, the screw which holds the protractor in place must be removed if one wishes to use a declination numerically greater than 27° with a latitude less than 45°.

As is the case with all graphical methods, (and this is just a mechanized nomogram), the accuracy to be obtained is limited. For ordinary navigation purposes, it is adequate.

The vinylite sheet carries a graphical presentation of the approximate declination of the sun and another of the approximate equation of time, allowing one to use the instrument to find the azimuth of the sun, even though an almanac is not at hand.

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EDITORIAL NOTE: The Azimuth Computer of JOSEPH HILSENRATH and SAMUEL GLOBE, of the U. S. Navy, was first put on the market in Dec. 1944, when it was reviewed in *Boat and Equipment News*, v. 7, p. 56. It was also reviewed in *The Rudder*, v. 61, May 1945, p. 42–43, and *Yachting*, v. 78, Nov. 1945, p. 71. On Oct. 6, 1946 it was patented (no. 2408776) and the printed description in *U. S. Specifications and Drawings* occupies 6 columns and 2 plates (a third plate having been canceled).

3. SAMUEL HERRICK, "Instrumental solutions in celestial navigation," *Navigation*, v. 1, no. 2, June 1946, p. 22–27.

Quotation: "This paper attempts to survey a selection of the mechanical devices already in existence and to suggest possible developments along the lines indicated by: (1) devices that give altitude and azimuth; (2) devices that give position; (3) devices that combine observation and reduction; (4) the need for increased accuracy." The illustrations are (a) The Hagner Position Finder and Sun Compass; (b) A universal instrument for projecting lines of position onto a chart; (c) The Kaster spherant; (d) An instrument for the determination of latitude and longitude from simultaneous observations of two stars. There is a bibliography of eight references.

4. I. ĬA. Akushskiĭ, "Numerical solution of the Dirichlet equation with the aid of perforated card machines," Akad. N., SSSR, (*Dok.*) *C.R.*, v. 52, 1946, p. 375–378. See *Math. Rev.*, v. 8, 1947, p. 288, P. W. Ketchum.

5. Neil R. Bartlett, "A punched-card technique for computing means, standard deviations, and the product-moment correlation coefficient and for listing scattergrams," *Science*, v. 104, 18 Oct. 1946, p. 374–375.

6. S. Bergman, "Construction of a complete set of solutions of linear partial differential equations in two variables by use of punch card machines," *Quart. Appl. Math.*, v. 4, 1946, p. 233–245.

7. W. J. Eckert, "Punched-card techniques and their application to scientific problems," *Jn. Chem. Educ.*, v. 24, Feb. 1947, p. 54–57.

8. Paul Herget, "Numerical integration with punched cards," *Astron. Jn.*, v. 52, 1946, p. 115–117.

Development of ideas set forth by W. J. Eckert, *Astron. Jn.*, v. 44, 1935, p. 177–182, "The computation of special perturbations by the punched card method," regarding the use of punched cards for integration, in the solution of the equations of motion of the three- or *n*-body problem. See *Math. Rev.*, v. 8, 1947, p. 289, Z. Kopal.

9. G. Kind-Schaad, "Lösung von Eigenwertproblemen mittels Lochkartenmaschinen" [Solution of characteristic value (Eigenwert) problems by punched card machines], *Schweizer Archiv f. angew. Wissen. u. Technik*, v. 13, June 1947, p. 161–168.

10. Gilbert W. King, "Some applications of punched-card methods in research problems in chemical physics," *Jn. Chem. Educ.*, v. 24, Feb. 1947, p. 61–64.

11. H. W. Renner, "Solving simultaneous equations through the use of IBM electric punched card accounting machines," 6 p. + 2 plates, a personal paper procurable at IBM, Endicott, N. Y.

12. Wm. A. Reynolds, "A prepunched master deck for the computation of square roots on IBM electrical accounting equipment," *Psychometrika*, v. 11, 1946, p. 223–237 + 1 folding plate.

Quotation: "Such a deck is valuable in constructing mathematical tables which involve square roots or in obtaining standard deviations in connection with computing correlation coefficients."

13. P. A. Shaffer, Jr., Verner Schomaker & Linus Pauling, "The use of punched cards in molecular structure determinations, I. Crystal structure calculations," *Jn. Chem. Physics*, v. 14, Nov. 1946, p. 648–658.

14. P. A. Shaffer, Jr., Verner Schomaker, & Linus Pauling, "The use of punched cards in molecular structure determinations, II. Electron diffraction calculations," *Jn. Chem. Physics*, v. 14, Nov. 1946, p. 659–664.

15. Clifford E. Berry & J. C. Pemberton, "A twelve-equation computing instrument," *Instruments*, v. 19, 1946, p. 396–398.

See the article by Berry, Wilcox, Rock & Washburn, "A computer for solving linear simultaneous equations," reviewed by D. H. L., *MTAC*, v. 2, p. 222–223.

16. I͡u. G. Tolstov, "Novyĭ elektricheskiĭ apparat dli͡a harmonicheskogo analiza i sinteza" [A new electrical apparatus for harmonic analysis and synthesis], Akad. N., SSSR, *Izvestii͡a, Otdelenie tekhnicheskikh N.*, Apr. 1946, no. 3, p. 389–400. See *Math. Rev.*, v. 8, 1947, p. 287, H. B. Curry.

17. I. S. Bruk, "A mechanical device for the approximate solution of the Poisson-Laplace equations," Akad. N., SSSR (*Dok.*) *C.R.*, v. 53, 1946, p. 311–312. See *Math. Rev.*, v. 8, 1947, p. 288, S. H. C.

18. I. S. Bruk, "A device for the solution of ordinary differential equations," Akad. N., SSSR (*Dok.*) *C.R.*, v. 53, 1946, p. 523–526. See *Math. Rev.*, v. 8, 1947, p. 288, S. H. C.

19. R. Fürth & R. W. Pringle, "A photo-electric Fourier transformer," *Phil. Mag.*, s. 7, v. 37, 1946, p. 1–13. See *Math. Rev.*, v. 8, 1947, p. 287–288, S. H. C. See also *MTAC*, v. 2, p. 89.

NOTES

80. Certain Gear-Ratio Tables.—We have previously referred to gear ratios in *MTAC*, v. 1, p. 21–23, 88, 92, 143, 324, 326–329, 430. In the tenth edition of their *Formulas in Gearing*, 1929, the Brown & Sharpe Mfg. Co. first published their 6D table, p. 239–243, Logarithm of Gear Ratios N/D, $N \leqslant 100$, $D \leqslant 100$, $\frac{24}{100} < N/D < \frac{100}{24}$. It was not until the eleventh edition, 1933, p. 227, that to the title of these tables is appended the footnote, "Wingquist's Tables (American Machinist)." This footnote is quoted by FMR, *Index*, p. 22. On appealing to Henry D. Sharpe, President of the Company, and Chancellor of Brown University, he kindly furnished the following details:

Erik Wingquist, in *American Machinist*, v. 43, 1915, p. 1080–1083, 1114–1118, published a 7D table of log N/D, for gear-ratios $\frac{30}{100} < N/D < \frac{100}{30}$. "Apparently we had used these tables in making hob-sheet calculations of the gearing for backing-off lathes. The tables, however, stopped at 30 teeth whereas we had to use change gears with as few as 24 teeth. When we decided to include a table of logarithms of gear ratios in the *Formulas*, it was also decided that the table should go down to 24:100. Accordingly our Mr. L. R. Mayo made the necessary revisions of Wingquist's tables to interpose and add all the new ratios involved with pinions having numbers of teeth between 24 and 30."

Hence the table in *Formulas*, by two authors, Wingquist & Mayo, is both an expansion and abridgment of Wingquist's table. Thus there is call for revision of the FMR entry in order accurately to present all that is here involved.

R. C. A.

81. GUIDE (*MTAC*, no. 7), SUPPL. 6 (for Suppl. 1–5 see v. 1, p. 403, v. 2, p. 59, 92, 190f., 224).—S. P. GLAZENAP, *Matematicheskie i Astronomicheskie Tablišy*, Leningrad, 1932, p. 103: 4D tables of $C[(2x/\pi)^{\frac{1}{2}}]$, and $S[(2x/\pi)^{\frac{1}{2}}]$, for $x = .04(.04)1, .5(.5)50$. There are two last-figure errors, namely: in S for $x = 1.5$, and in C for $x = 11.5$. These are corrected in JAHNKE & EMDE, *Tables of Functions*, 1945, p. 35.

N. R. JØRGENSEN, *Undersøgelser over Frequensflader og Korrelation*, Diss. Copenhagen, 1916. T. II, p. 153–156, $\log f_n(t) = \log [t^{-n}I_n(t)]$, for $n = 0(1)11$, $t = [0(.1)6; 7D]$, with first differences; the last figure is unreliable. This appears to be the only published table of this kind.

While it was noted in *MTAC*, v. 1, p. 253, 305, that B. A. SMITH first tabulated and named Michell function in a paper on "Arched dams," a reference was given only to A.S.C.E., *Trans.*, v. 83 for years 1919–1920 (with title page dated 1921), p. 2027–2077, table on p. 2052–2055. The same table was, however, given in A.S.C.E., *Papers and Discussions*, v. 46, 1920, p. 400–403.

R. C. A. & MURLAN S. CORRINGTON

82. HERMAN HOLLERITH.—The American Council of Learned Societies and Charles Scribner's Sons have graciously granted us special permission to quote the following sketch of Hollerith written by the late Professor W. F. Willcox of Cornell University and published in *Dictionary of American Biography*, v. 21, New York, 1944, p. 415–416:

Hollerith, Herman (Feb. 29, 1860–Nov. 17, 1929), inventor of tabulating machines, was born in Buffalo, N. Y., the son of George and Franciska (Brunn) Hollerith. After preliminary schooling he attended the School of Mines of Columbia University and was graduated in 1879. Immediately thereafter he became an assistant to his teacher, William Petit Trowbridge [q.v.], in the Census of 1880. He worked on the statistics of manufacturers and prepared an article, "Report on the Statistics of Steam- and Water-Power Used in the Manufacture of Iron and Steel," for the *Report on Power and Machinery Employed in Manufactures* (Census Office, Department of the Interior, 1888). His work on the census brought him into contact with Dr. John Shaw Billings [q.v.], from whom came the suggestion of Hollerith's main invention. In a letter to a friend written nearly forty years later he described the origin of the idea: "One evening at Dr. B's tea table he said to me, 'There ought to be a machine for doing the purely mechanical work of tabulating population and similar statistics.' " Hollerith thought the problem could be solved and later offered Billings a share in the project.

In 1882 he went to the Massachusetts Institute of Technology, as instructor in mechanical engineering. He disliked teaching, however, and after a year moved to St. Louis, Mo., where he experimented on electromagnetically operated air-brakes and other types of brakes for railroads. From 1884 to 1890 he was attached to the Patent Office in Washington, D. C. During these years he worked on the problem of perfecting mechanical aids in tabulating statistical information. By the time the Census of 1890 was to be taken he had invented machines that would record statistical items, by a system of punched holes in a non-conducting material, and would also count those items by means of an electric current passed through the holes

identically placed. The system was given trial in tabulating mortality statistics in Baltimore, and in compiling similar data in New Jersey and New York City. In competition with two alternative methods of tabulation, it was chosen for use in compiling the Census of 1890. It did a sample piece of work in less than half the time required by the other systems, and the commission estimated that in dealing with the returns expected at the approaching census the new machine would reduce the labor days by more than two-thirds. Subsequently the machines were improved by the addition of a mechanical feeding device. In 1890 the Franklin Institute of Philadelphia, reporting that Hollerith had made the outstanding invention of the year, gave him its highest award, the Elliott Cresson medal.

The Hollerith machines were used in 1891 in recording the census returns in Canada, Norway, and Austria. Although they revolutionized statistical technique, American scholars gave little attention to them at the outset, probably because statistical interpretation had not been carried as far in the United States as elsewhere. But in Europe technical articles about their value appeared in England, France, Germany, Austria, and Italy. Hollerith attended the Berne session of the International Statistical Institute in 1895 and commented upon a paper by an Austrian member. Between 1890 and 1900 the machines were successfully adapted to handle types of mass enquiries in which addition was an element, and thus they could be used in tabulating railroad freight statistics and the data assembled in the agricultural census.

In 1896 Hollerith organized the Tabulating Machine Company, incorporated in New York, to manufacture the machines and to sell the cards used with them. In 1911 that company was consolidated with the Computing Scale Company of America and the International Time Recording Company of New York to become the Computing-Tabulating-Recording Company, later known as the International Business Machines Corporation, of which Hollerith was retained as consulting engineer until 1921.

Hollerith died at his home in Washington, of heart disease, at the age of sixty-nine. . . . During his lifetime Hollerith received more than thirty patents from the United States Government, as well as many from foreign governments.

EDITORIAL NOTE: Hollerith received from Columbia University the E.M. degree in 1879 and the Ph.D. degree in 1890.

83. INTERPOLATION OF FUNCTIONS TABULATED AT FIXED POINTS.—In the recent note "The checking of functions tabulated at certain fractional points," the writer should have mentioned that those tabulated coefficients of f_k, say $A_k^{(n)}$, which give the $(n - 1)$th divided difference for n particular points, $x_0, x_1, \cdots, x_{n-1}$ can also be used for interpolation according to a generalization of the scheme in W. J. TAYLOR, "Method of Lagrangian curvilinear interpolation," NBS, *Jn. Res.*, v. 35, 1945, RP 1667, p. 151–155. For, setting $a_k = A_k^{(n)}/(x - x_k)$, Taylor's scheme is essentially

$$f \sim \Big(\sum_{k=0}^{n-1} a_k f_k\Big)\Big/\sum_{k=0}^{n-1} a_k.$$

This generalization of Taylor's scheme can always make use of the coefficients of the $(n - 1)$th divided difference for any set of fixed irregularly-

spaced points. In particular, the coefficients for the divided differences given in the author's "Coefficients for facilitating the use of the Gaussian quadrature formula," *Jn. Math. Phys.*, 1946, p. 246, and in "Tables for facilitating the use of Chebyshev's quadrature formula" (forthcoming), can also be used to interpolate for any function which is given at the zeros of either the Legendre or Chebyshev polynomials respectively.

HERBERT E. SALZER

NBSCL

84. THE NATIONAL APPLIED MATHEMATICS LABORATORIES.—After prolonged study and many conferences the National Bureau of Standards (NBS) started on July 1, 1947, to set up The National Applied Mathematics Laboratories (NAML) which, according to the plans set forth in *A Prospectus* (February, 1947, x, 46 p., with a Foreword by E. U. CONDON, Director of NBS), will not have its full initial personnel until 1949. We quote from this document. Applied mathematics is on the threshold of revolutionary developments which will permit numerical answers to physical problems to be obtained at hitherto undreamed of speeds. A strong, easily accessible, federal applied mathematics center, operating with low overhead costs, providing economical but competent computational and consulting services, and performing forward-looking research in the newer methods of applied mathematics, is a necessity in the national research program.

The work of the NAML, with Dr. JOHN H. CURTISS as Chief, is to be carried on with the guidance of a committee of representatives of interested outside groups, called the Applied Mathematics Executive Council. The NAML consists of four major units, as follows:

I. *The Institute for Numerical Analysis*, located at the University of California in Los Angeles, and starting operations in 1948. The present staff includes the following members: Mr. JOHN TODD, late of King's College, University of London; Dr. HARRY D. HUSKEY, recently of the National Physical Laboratory, Teddington; and Professor OTTO SZÁSZ, recently of the University of Cincinnati. Professor D. R. HARTREE of the University of Cambridge has been appointed Acting chief of the Institute for the duration of his stay in this country, June–September 1948.

II. *The Computation Laboratory*, located in New York or Washington; Dr. A. N. LOWAN is now Chief. In the future we shall refer to this group by the symbol NBSCL rather than NBSMTP.

III. *Statistical Engineering Laboratory* at the NBS, with Dr. CHURCHILL EISENHART as Chief. This is a statistical consulting service specializing in the application of modern statistical inference to the physical and engineering sciences.

IV. *Machine Development Laboratory*, Dr. E. W. CANNON, Chief, with A. Mathematics Group; B. Electronics Group, in the NBS. This is a Laboratory devoted to the development of automatic digital computing machinery, and its staff edits the department ACM in *MTAC*. Units II–IV are now functioning.

The general plan envisions a stabilization of operations within two years at a level of about 94 man-years and $532,000 annually. The estimated capi-

tal cost of automatic computing equipment, not included in these figures, is about $700,000.

The following NAML publication (Nov. 1947, 25p. 20.3 × 26.7 cm.) is an interesting one: *Activities in Applied Mathematics 1946–1947*.

85. NAML: THE COMPUTATION LABORATORY.—At this Laboratory there is now a staff of 60 people. The bound volumes of mathematical tables, issued from the NYMTP later NBSMTP, but now NBSCL, apart from those published by the Columbia University Press, are now distributed by the Superintendent of Documents, Washington, D. C. In the case of each of these volumes there were about 1500 copies in the edition. *Tables of the Exponential Function* e^x (1939) has long been out of print but a new edition is now in the press at the Government Printing Office. The edition of *Tables of Probability Functions* (2 v., 1941–1942) is nearly exhausted, but it is planned that these and other volumes shall be similarly reprinted.

Of volumes of the NBSCL published by the Columbia University Press the first editions consisted of approximately 800 copies. A second edition of *Table of The Bessel Functions* $J_0(z)$ *and* $J_1(z)$ (1943) is reviewed elsewhere in this issue, and a second edition of *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments* (1943) has been published.

The Superintendent of Documents, Washington, D. C., lists for sale 36 mathematical tables emanating from the NBSCL and its predecessors. Of these, 19, nos. M18–36, are shorter tables, mostly reprinted from the M. I. T. *Jn. Math. Physics*. This series is being continued, with a designation yet to be determined; its issues are not to be for sale. There is also a new publication of the NBS, *The Applied Mathematics Series*, containing mathematical tables, manuals and studies, by members of the mathematical laboratories of the NBS. More than 30,000 of the NBSCL tables are now in use, and the annual demand runs at the rate of about 3500 volumes.

86. RUSSIA AND MATHEMATICS.—In *Soviet News*, Soviet Embassy in London, no. 1542, Sept. 26, 1946 is an article by S. VAVILOV, president of the Soviet Academy of Sciences, entitled "Our five-year plan for science." The section on "Mathematics as a theoretical weapon" is as follows:

Mathematics, which is of vital importance to natural science, to technique, and to such social sciences as economics, is directly linked up with problems of philosophy and logic. Much of our five-year plan for mathematics is directed towards assisting other sciences. For example, we are stressing questions of the theory of probability—particularly those bearing on the interpretation of observations and research on partial differential equations, particularly those associated with what may be termed "machine mathematics," that is, the solution of mathematical problems with the help of calculating devices.

Calculating machines have been known for centuries. But never has "machine mathematics" reached such scope as at the present time. New calculators devised on electrical principles make it possible to solve extremely complex mathematical problems connected with technique and the various branches of natural science. So important do we think this side of mathematics, that we are proposing in the immediate future to devote to it a special institute.

However, machines can never displace mathematical thought. A characteristic of mathematical thought is its boldness, its imaginative power. Such creative mathematics, which at times finds no immediate application in technical science, has always found fertile ground in our Academy; and its development must continue on a broad scale. Such subjects as non-Euclidean geometry, the tensor calculus and the theory of groups, which seem to be abstract studies absolutely cut off from life and from reality, nevertheless suddenly assume a decisive significance at definite stages of scientific development.

This explains the inclusion in the Academy's plan of the problems of the theory of numbers, abstract algebra, topology and mathematical logic.

QUERIES

24. INTEGRAL OF A STRUVE FUNCTION.—In a problem of diffraction the following integral comes up: $\int_0^x H_0(t)dt$. Where may I find a published table of this integral for the range $x = [0(.1)10; 6D]$?

C. W. HORTON

Defense Research Laboratory
University of Texas

EDITORIAL NOTE: Since we received this Query for publication there came from Dr. JOHN W. WRENCH, JR., a copy of a table of $\int_0^x H_0(t)dt$, $x = [.1(.1)10; 8D]$. This was obtained by numerical integration of the values of $H_0(x)$ tabulated in G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge and New York, 1944, p. 666–684. For integral values of the upper limit the integral was calculated to about 15D by infinite series. This check on certain of the entries indicates that the maximum error in any entry should not exceed 0.6×10^{-8}.

25. RUSSIAN BESSEL FUNCTION TABLES.—According to an advertisement in *Matematicheskiĭ Sbornik*, v. 51, no. 3, 12 June 1941, the Mathematical Institute of the USSR Academy of Sciences was at this time in 1941 about to publish a volume entitled *Tablifsy Besselevykh Funkfsiĭ Mnimogo Argumenta* [Tables of Bessel Functions with Complex Argument]. It was announced that the volume contained about 400 pages and was to cost 25 roubles. The Library of Brown University has long but vainly tried to procure a copy. Can any reader report on the ownership of this volume of tables by any individual or library, or on a copy for sale?

R. C. A.

QUERIES—REPLIES

33. PORTRAITS AND BIOGRAPHIES OF BRITISH MATHEMATICAL TABLE MAKERS (Q21, v. 2, p. 286).—Herewith are submitted references to biographical material concerning four of these table makers. (a) PETER BARLOW: Amer. Acad. Arts & Sci., *Proc.*, v. 6, 1866, p. 15–16; Inst. Civil Engin., *Proc.*, v. 22, 1863, p. 615–618; R.A.S., *Mo. Not.*, v. 23, 1863, p. 127–128; R. Soc. London, *Proc.*, v. 12, 1863, p. xxxiii–xxxiv. Sir HUMPHRY DAVY, *Six Discourses*, London, 1827, p. 111–115, also in H. DAVY, *Coll. Works*, v. 7, London, 1840, p. 76, 83–89. (b) RICHARD FARLEY: R.A.S., *Mo. Not.*, v. 40, 1880, p. 192–194. (c) HIRSCH FILIPOWSKI: *Jewish Encyclopedia*, v. 5, New York, 1903, p. 383 (M. BEER). (d) PETER GRAY: R.A.S., *Mo. Not.*, v. 48, 1888, p. 163–165.

JEKUTHIEL GINSBURG

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HENRY BRIGGS: Since KARL PEARSON made his inquiries (on my behalf) about a portrait, in 1924, I have made searches at the British Museum and the National Portrait Gallery and elsewhere, and am forced to the conclusion that no portrait exists. JAMES DODSON: Inst. Actuaries, *Jn. and Assurance Mag.*, v. 14, 1867–68, p. 341 (A. DE MORGAN).

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EDITORIAL NOTE: Of those mentioned above, the *Dict. Nat. Biog.* contains sketches of the following: (i) BARLOW, v. 3, 1885, p. 222–224 (A. M. CLERKE); (ii) BRIGGS, v. 6, 1886, p. 326–327 (T. WHITTAKER); (iii) DODSON, v. 15, 1888, p. 174–175 (G. J. GRAY); (iv) GRAY, v. 23, 1890, p. 16 (G. GOODWIN); there are also sketches of Gray in C. WALFORD, *Insurance Cyclopaedia*, London, v. 5, 1878, p. 540–541, in Inst. Actuaries, *Jn.*, v. 26, 1889, p. 301–302, 406, and in *Intern. Insurance Encyclopedia*, New York, 1910, p. 329–330. For Dodson, see also *Int. Ins. Encycl.*, p. 227, and WALFORD, *Ins. C.*, v. 2, 1873, p. 390–391. There are also sketches of Filipowski, mathematician, linguist, editor, in Walford, *Ins. C.*, v. 3, 1874, p. 296–297 (not an F. R. S. as here stated), in *Jüdisches Lexikon*, v. 2, Berlin, 1928 (KARL KARPELESZ), and in *Universal Jewish Encyclopedia*, New York, v. 4, 1941, where the name is given as "Zebi Hirsch Filipowski." In his *A Table of Anti-Logarithms . . .*, London, 1849, and second ed. rev. and corrected, London, 1851, the name appears as Herschell E. Filipowski. Herschell Filipowski is also the form of name used in 1857 in his English edition of Napier's *Wonderful Canon*. Hence Herschell E. Filipowski (used also by Walford and by the *British Museum Catalogue*) appears to be the proper form of his name in connection with mathematical publications. In his edition of F. BAILY, *The Doctrine of Life-Annuities and Assurances*, London, 2 v., 1864–1866, the name is simply H. Filipowski. We have not discovered for what name the E. may stand. Herschell, Hersh and Hirsch are all names, meaning "deer" or "stag," for which Zebi is the Hebrew word. The repetition Zebi Hirsch is common among Jews (like Abba Father in the Bible).

CORRIGENDA

V. 2, p. 336, l. 9, *for* 348 *read* 384; p. 356, l. −1, *for* by the relays, *read* by the same relays; p. 368, l. 10, *for* 1940, *read* 1946; p. 375, l. −1, *for* 342, *read* 242.